Introduction

This document contains a list of mathematical problems that you — the applicant for the master program *Applied Mathematics for Network and Data Sciences* at the University Mittweida — should try to solve. We recommend attempting to solve in a first step as many problems as you can without consulting any text books, web pages, or friends. In a second step you should search for definitions and notions that you do not know so that you become able to solve some additional problems.

Please observe:

1. You should verify the solution *after* you have worked on *all* problems.
2. The problems are intended to check your knowledge by self-evaluation using the provided solutions. You should never send your solutions to the University Mittweida. (They will be ignored.)

Problems

Exercise 1 *Prove that for any real positive x, the inequality*

$$x + \frac{1}{x} \geq 2$$

is satisfied.

Exercise 2 *Prove by induction that the equality*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

is satisfied for any nonnegative integer n and any $x, y \in \mathbb{R}$.

Exercise 3 *Show that there are infinitely many primes.*
Hint*: You can assume that the unique prime factorization theorem is known.*

Exercise 4 *Let n be a positive integer. Find the value of the double sum*

$$\sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^j k.$$

Exercise 5 *Let G be a group and $x, y, z \in G$. Prove that $xz = yz$ implies $x = y$.*

Exercise 6 *Let A, B be sets, $f : A \to B$ a mapping, and $Y, Z \subseteq B$. Prove*

$$f^{-1}(Y \cup Z) = f^{-1}(Y) \cup f^{-1}(Z).$$

Hint*: The set $f^{-1}(Y)$ is defined as the set of all $x \in A$ such that $f(x) \in Y$.*

Exercise 7 *Let n be a positive integer and $A = (a_{ij})$ a real $n \times n$ matrix with determinant $\det A = \delta$. Find $\det(\alpha A)$ for any fixed $\alpha \in \mathbb{R}$.*

Exercise 8 *Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ be linearly independent vectors. Give a formula for the volume of the parallelepiped spanned by $\mathbf{a}, \mathbf{b}, \mathbf{c}$.*

Exercise 9 *Let $k > 0$ be an integer. Show that*

$$x \equiv y \quad \Longleftrightarrow \quad k \text{ devides } x - y$$

defines an equivalence relation on $\mathbb{Z}$.

Exercise 10 *Prove that $\sqrt{3}$ is irrational.*

Exercise 11 *Assume $A_1, \ldots, A_n$ are random events. Show that*

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \Pr(A_i).$$

Exercise 12 *Let m be a positive integer, $n = m + 2$, A a real $m \times n$ matrix of rank $m - 1$, and $\mathbf{b} \in \mathbb{R}^m$ a fixed vector. Assume that the rank of the matrix $(A|\mathbf{b})$ obtained from A by adding the column $\mathbf{b}$ is equal to the rank of A. Describe the solution space of the system of linear equations $A\mathbf{x} = \mathbf{b}$.*

Exercise 13 *Assume the two real random variables X and Y are independent and uniformly distributed in $[0, 1]$. Find the probability $\Pr(\{|X - Y| < \frac{1}{2}\})$.*

Exercise 14 *Consider the random experiment of throwing two dice. Define the random variable X as the maximum of the two resulting numbers. Calculate the expectation $\mathbb{E}X$.*

Exercise 15 *Prove that*

$$\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right)\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + - \cdots\right) = 1.$$

Exercise 16 *Let n be a positive integer. Find the number of different nonnegative integer solutions of*

$$x + y + z = n.$$

Exercise 17 *Let A be a nonempty set. Assume that there exists an injective mapping $f : A \to B$ into a set B. Show that there exists a surjective mapping from B to A.*

Exercise 18 *Prove that the Cartesian product of two countable sets is countable.*

Exercise 19 *What are the accumulation points of the sequence*

$$x_n = \left(-1 - \frac{1}{n}\right)^n ?$$

Exercise 20 *Let* $m > 1$ *be an integer that is not a prime. Prove that* $\mathbb{Z}/m\mathbb{Z}$ *is not a field.*

Exercise 21 *Calculate* $123456^{78} \mod 5$.

Exercise 22 *Let* n *be a positive integer. How many different local maximum points has the real function*

$$f(x) = \prod_{k=0}^{2n+1} (x-k)?$$

Exercise 23 *Let* G *be a simple undirected graph with* n *vertices and* m *edges such that* $m \geq n$. *Show that* G *has a cycle.*

Exercise 24 *Find the value of the following integral:*

$$\int_0^1 \int_0^x \int_0^y y\, z \; dz\, dy\, dx$$

Exercise 25 *Solve the differential equation*

$$\frac{dy}{dt} = y + t$$

for the initial value $y(0) = 1$.