

## Some Advice for the Evaluation

The presented solutions are possible ways to solve the problems. In most cases there is more than one way to attack them. In case you have a completely different proof than the one given here, you might seek for advice from a local expert.

If you could independently solve more than 12 problems, than you have the mathematical background to apply for the master program *Applied Mathematics for Network and Data Sciences* at the University Mittweida. You are clearly better prepared when you have solved more problems.

You should definitely not apply for the master program if you could solve less than eight problems completely.

## Solutions to the Problems

**Exercise 1** Prove that for any real positive  $x$ , the inequality

$$x + \frac{1}{x} \geq 2$$

is satisfied.

**Solution:** We rewrite the given inequality as

$$x^2 - 2x + 1 \geq 0.$$

Factoring the polynomial yields

$$(x - 1)^2 \geq 0,$$

which is true as any square of a real number is a nonnegative real number.

**Exercise 2** Prove by induction that the equality

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

is satisfied for any nonnegative integer  $n$  and any  $x, y \in \mathbb{R}$ .

**Solution:**

*Induction base:* For  $n = 0$ , we obtain

$$(x + y)^0 = 1 = \sum_{k=0}^0 \binom{0}{k} x^k y^{0-k}. \quad \checkmark$$

*Induction hypothesis:* Assume that the statement is true for a fixed number  $n \in \mathbb{N}$ :

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

*Induction step:* The calculation

$$\begin{aligned}
 (x+y)^{n+1} &= (x+y)(x+y)^n \\
 &= (x+y) \sum_{k=0}^n \binom{n}{k} x^k y^{n-k} \\
 &= \sum_{k=0}^n \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=0}^n \binom{n}{k} x^k y^{n+1-k} \\
 &= \sum_{k=1}^{n+1} \binom{n}{k-1} x^k y^{n+1-k} + \sum_{k=0}^{n+1} \binom{n}{k} x^k y^{n+1-k} \\
 &= \sum_{k=0}^{n+1} \left[ \binom{n}{k-1} + \binom{n}{k} \right] x^k y^{n+1-k} \\
 &= \sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k}
 \end{aligned}$$

shows that the statement is true for any nonnegative integer.

**Exercise 3** Show that there are infinitely many primes.

*Hint:* You can assume that the unique prime factorization theorem is known.

**Solution:** We perform the proof by contradiction. Assume that there only finitely many primes. Let  $P = \{p_1, p_2, \dots, p_n\}$  be the set of all primes. Now consider the number

$$x = p_1 \cdot p_2 \cdots p_n + 1.$$

The unique prime factorization theorem states that any integer greater than 1 is a product of (uniquely determined) primes. Consequently,  $x$  should be divisible by a prime from  $P$ . However, the remainder is always equal to 1 when we divide  $x$  by any  $p_i \in P$ , which is a contradiction. We conclude that there are infinitely many primes.

**Exercise 4** Let  $n$  be a positive integer. Find the value of the double sum

$$\sum_{k=0}^n \sum_{j=0}^k (-1)^j k.$$

**Solution:** We have

$$\sum_{k=0}^n \sum_{j=0}^k (-1)^j k = \sum_{k=0}^n k \sum_{j=0}^k (-1)^j.$$

The inner sum is equal to 1 if  $k$  is even and equal to 0 for odd  $k$ , which implies

$$\begin{aligned} \sum_{k=0}^n \sum_{j=0}^k (-1)^j k &= \sum_{\substack{k=0 \\ k \text{ even}}}^n k \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} 2k \\ &= 0 + 2 + 4 + \cdots + 2 \lfloor \frac{n}{2} \rfloor \\ &= \lfloor \frac{n}{2} \rfloor \left( \lfloor \frac{n}{2} \rfloor + 1 \right). \end{aligned}$$

**Exercise 5** Let  $G$  be a group and  $x, y, z \in G$ . Prove that  $xz = yz$  implies  $x = y$ .

**Solution:** By multiplication of  $xz = yz$  with  $z^{-1}$  from right, we obtain

$$xzz^{-1} = yzz^{-1}.$$

Now we use  $zz^{-1} = e$  (the neutral element), which gives

$$xe = ye$$

and hence  $x = y$ .

**Exercise 6** Let  $A, B$  be sets,  $f : A \rightarrow B$  a mapping, and  $Y, Z \subseteq B$ . Prove

$$f^{-1}(Y \cup Z) = f^{-1}(Y) \cup f^{-1}(Z).$$

Hint: The set  $f^{-1}(Y)$  is defined as the set of all  $x \in A$  such that  $f(x) \in Y$ .

**Solution:** We apply the definition of  $f^{-1}$  to obtain

$$\begin{aligned} f^{-1}(Y \cup Z) &= \{x \in A \mid f(x) \in Y \cup Z\} \\ &= \{x \in A \mid f(x) \in Y \text{ or } f(x) \in Z\} \\ &= \{(x \in A \text{ and } f(x) \in Y) \text{ or } (x \in A \text{ and } f(x) \in Z)\} \\ &= \{x \in A \mid f(x) \in Y\} \cup \{x \in A \mid f(x) \in Z\} \\ &= f^{-1}(Y) \cup f^{-1}(Z). \end{aligned}$$

**Exercise 7** Let  $n$  be a positive integer and  $A = (a_{ij})$  a real  $n \times n$  matrix with determinant  $\det A = \delta$ . Find  $\det(\alpha A)$  for any fixed  $\alpha \in \mathbb{R}$ .

**Solution:** As the determinant is linear in any column vector we obtain

$$\det(\alpha A) = \alpha^n \det A.$$

**Exercise 8** Let  $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$  be linearly independent vectors. Give a formula for the volume of the parallelepiped spanned by  $\mathbf{a}, \mathbf{b}, \mathbf{c}$ .

**Solution:** The volume is given by the absolute value of the scalar triple product (or box product) of the three vectors:

$$V = |(\mathbf{a}, \mathbf{b}, \mathbf{c})| = |\det(\mathbf{a}, \mathbf{b}, \mathbf{c})| = |\langle \mathbf{a}, \mathbf{b} \times \mathbf{c} \rangle|$$

Here  $\langle \mathbf{x}, \mathbf{y} \rangle$  denotes the inner product (scalar product) of two vectors  $\mathbf{x}$  and  $\mathbf{y}$ , which is sometimes differently denoted by  $\mathbf{x} \cdot \mathbf{y}$ .

**Exercise 9** Let  $k > 0$  be an integer. Show that

$$x \equiv y \iff k \text{ divides } x - y$$

defines an equivalence relation on  $\mathbb{Z}$ .

**Solution:** To show that a relation is an equivalence relation we have to prove that it is reflexive, symmetric, and transitive.

- Any positive integer divides zero. Consequently, we have  $x \equiv x$  for any  $x \in \mathbb{Z}$  which implies that  $\equiv$  is a reflexive relation.
- If  $k$  divides  $x$ , then  $k$  also divides  $-x$ , which implies that  $x \equiv y$  if and only if  $y \equiv x$ . Hence we have symmetry.
- Now assume that for three integers  $x, y, z$ , we have  $x \equiv y$  and  $y \equiv z$ . Then there are integers  $a, b$  such that  $x - y = ka$  and  $y - z = kb$ . Then we conclude

$$x - z = (x - y) + (y - z) = ka + kb = k(a + b),$$

which shows that  $x - z$  is a multiple of  $k$  implying that  $k$  divides  $x - z$  and hence  $x \equiv z$ . This shows that the relation is transitive.

**Exercise 10** Prove that  $\sqrt{3}$  is irrational.

**Solution:** We show this by contradiction. Assume that  $\sqrt{3}$  is rational. Then there exist positive integers  $p$  and  $q$  satisfying

$$\sqrt{3} = \frac{p}{q}.$$

We can assume that all common prime factors of  $p$  and  $q$  are canceled so that we have the shortest possible form of this fraction, which implies that 3 cannot be a divisor of  $p$  and  $q$ . Now we square the equation, which gives

$$3 = \frac{p^2}{q^2}$$

or

$$p^2 = 3q^2. \tag{1}$$

We abbreviate  $q^2 = n$ , which yields  $p^2 = 3n$ . Assume that we can find an integer  $m$  such that  $p = 3m + 1$ . Then we find  $p^2 = 9m^2 + 6m + 1$ . However as  $p^2$  is a multiple of 3, there cannot exist such  $m$ . In the same way we can show that  $p = 3m + 2$  is impossible, which implies that  $p = 3m$  for a suitably selected integer  $m$ . Squaring this relation yields  $p^2 = 9m^2$ . Substituting  $p^2$  in Equation (1) gives

$$9m^2 = 3q^2$$

and hence

$$3m^2 = q^2,$$

which shows that  $q^2$  and hence (with the same argumentation as above) also  $q$  is divisible by 3. However, that would mean that we can cancel the common factor 3 in the fraction  $p/q$ , which is a contradiction to our assumption that this fraction has the shortest possible form. Consequently there is no fraction (no rational number) that represents  $\sqrt{3}$  implying that  $\sqrt{3}$  is irrational.

**Exercise 11** Assume  $A_1, \dots, A_n$  are random events. Show that

$$\Pr\left(\bigcup_{i=1}^n A_i\right) \leq \sum_{i=1}^n \Pr(A_i).$$

**Solution:** We prove this statement by induction on  $n$ . For  $n = 1$ , there is nothing to prove as we have equality in this case. For  $n = 2$ , we obtain

$$\Pr(A_1 \cup A_2) = \Pr(A_1) + \Pr(A_2) - \Pr(A_1 \cap A_2) \leq \Pr(A_1) + \Pr(A_2). \quad (2)$$

Equation (2) can be easily derived from the axioms of a probability space. Now assume (as the induction hypothesis) that the statement,

$$\Pr\left(\bigcup_{i=1}^n A_i\right) \leq \sum_{i=1}^n \Pr(A_i), \quad (3)$$

is true for some  $n \geq 1$ . Then we can proceed as follows to obtain the desired statement:

$$\begin{aligned} \Pr\left(\bigcup_{i=1}^{n+1} A_i\right) &\stackrel{(2)}{\leq} \Pr\left(\bigcup_{i=1}^n A_i\right) + \Pr(A_{n+1}) \\ &\stackrel{(3)}{=} \sum_{i=1}^n \Pr(A_i) + \Pr(A_{n+1}) \\ &= \sum_{i=1}^{n+1} \Pr(A_i) \end{aligned}$$

**Exercise 12** Let  $m$  be a positive integer,  $n = m + 2$ ,  $A$  a real  $m \times n$  matrix of rank  $m - 1$ , and  $\mathbf{b} \in \mathbb{R}^m$  a fixed vector. Assume that the rank of the matrix  $(A|\mathbf{b})$  obtained from  $A$  by adding the column  $\mathbf{b}$  is equal to the rank of  $A$ . Describe the solution space of the system of linear equations  $\mathbf{Ax} = \mathbf{b}$ .

**Solution:** The condition

$$\text{rank}(A|\mathbf{b}) = \text{rank } A$$

ensures that the system has a solution. From  $\text{rank } A = m - 1$ , we know that there are  $m - 1$  linearly independent equations. As we have  $m + 2$  variables the solution contains 3 free parameters, which can be presented as follows

$$\mathbf{x} = \mathbf{a} + \alpha\mathbf{b} + \beta\mathbf{c} + \gamma\mathbf{d}, \quad \alpha, \beta, \gamma \in \mathbb{R}.$$

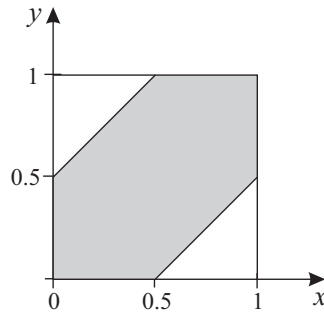
The vectors  $\mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^n$  span a 3-dimensional subspace of  $\mathbb{R}^n$ .

**Exercise 13** Assume the two real random variables  $X$  and  $Y$  are independent and uniformly distributed in  $[0, 1]$ . Find the probability  $\Pr(\{|X - Y| < \frac{1}{2}\})$ .

**Solution:**

$$\Pr\left(\left\{|X - Y| < \frac{1}{2}\right\}\right) = \Pr\left(\left\{-\frac{1}{2} < X - Y < \frac{1}{2}\right\}\right) = \frac{3}{4}$$

The following figure shows the range within the unit square where the inequalities are satisfied.



**Exercise 14** Consider the random experiment of throwing two dice. Define the random variable  $X$  as the maximum of the two resulting numbers. Calculate the expectation  $\mathbb{E}X$ .

**Solution:** By the definition of the expectation, we obtain

$$\mathbb{E}X = \sum_{k=1}^6 k \Pr(\{X = k\}),$$

which gives

$$\mathbb{E}X = 1 \cdot \frac{1}{36} + 2 \cdot \frac{3}{36} + 3 \cdot \frac{5}{36} + 4 \cdot \frac{7}{36} + 5 \cdot \frac{9}{36} + 6 \cdot \frac{11}{36} = \frac{161}{36}.$$

**Exercise 15** Prove that

$$\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots\right) \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots\right) = 1.$$

**Solution:** Observe that

$$\begin{aligned} \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right) &= \sum_{n \geq 0} \frac{1}{n!} = e \quad \text{and} \\ \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots\right) &= \sum_{n \geq 0} (-1)^n \frac{1}{n!} = e^{-1}. \end{aligned}$$

Now we easily obtain  $e \cdot e^{-1} = 1$ .

A different proof works as follows:

$$\begin{aligned} \left(\sum_{n \geq 0} \frac{1}{n!}\right) \left(\sum_{n \geq 0} (-1)^n \frac{1}{n!}\right) &= \sum_{n \geq 0} \sum_{k=0}^n (-1)^k \frac{1}{k!} \frac{1}{(n-k)!} \\ &= \sum_{n \geq 0} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} (-1)^k \\ &= \sum_{n \geq 0} \frac{1}{n!} \delta_{0,n} \\ &= 1, \end{aligned}$$

where  $\delta_{k,n}$  is defined by

$$\delta_{k,n} = \begin{cases} 1, & \text{if } n = k, \\ 0, & \text{if } n \neq k. \end{cases}$$

**Exercise 16** Let  $n$  be a positive integer. Find the number of different nonnegative integer solutions of

$$x + y + z = n.$$

**Solution:**

$$\binom{n+2}{2} = \frac{(n+1)(n+1)}{2}$$

**Exercise 17** Let  $A$  be a nonempty set. Assume that there exists an injective mapping  $f : A \rightarrow B$  into a set  $B$ . Show that there exists a surjective mapping from  $B$  to  $A$ .

**Solution:** An injective mapping  $f : A \rightarrow B$  defines in a natural way a bijective mapping  $f : A \rightarrow B'$ , where  $B'$  is the image of  $A$  under  $f$ . Now we define  $g : B \rightarrow A$  by

$$g(y) = \begin{cases} f^{-1}(y), & \text{if } y \in B', \\ a, & \text{if } y \in B \setminus B', \end{cases}$$

where  $a$  is an arbitrarily chosen element from  $A$ . Then we have for any  $x \in A$ ,  $g(f(x)) = x$ , which shows that  $g$  is surjective.

**Exercise 18** Prove that the Cartesian product of two countable sets is countable.

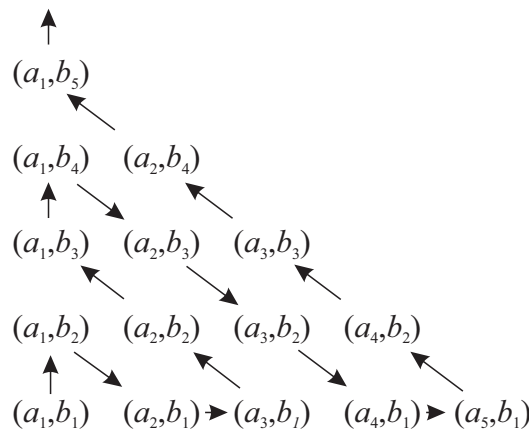
**Solution:** We use the same argument that shows that the set of rational numbers is countable. Without loss of generality, we can assume that the two sets are

$$A = \{a_1, a_2, a_3, \dots\} \quad \text{and} \quad B = \{b_1, b_2, b_3, \dots\}.$$

Then we have to show that

$$A \times B = \{(a_i, b_j) \mid a_i \in A \text{ and } b_j \in B\}$$

is a countable set, which means that we have to find a bijection  $f : A \times B \rightarrow \mathbb{N}$ . The assignment that defines  $f$  is presented in the following figure.



We define  $f((a_1, b_1)) = 1, f((a_1, b_2)) = 2, f((a_2, b_1)) = 3, f((a_3, b_1)) = 4, \dots$ . It is easy to verify that the given mapping is indeed bijective.

**Exercise 19** What are the accumulation points of the sequence

$$x_n = \left(-1 - \frac{1}{n}\right)^n ?$$

**Solution:**  $e$  and  $-e$

**Exercise 20** Let  $m > 1$  be an integer that is not a prime. Prove that  $\mathbb{Z}/m\mathbb{Z}$  is not a field.

**Solution:** Assume that  $m$  is composite. Then there are two integers  $r, s$  with  $r > 1$  and  $s > 1$  such that  $m = rs$ . But this implies that  $rs \equiv 0 \pmod{m}$  ( $r$  and  $s$  are divisors of zero). But in any field, the equation  $xy = 0$  implies that  $x = 0$  or  $y = 0$ . To show this, assume  $x \neq 0$ . Then we can multiply with  $x^{-1}$ , which yields  $x^{-1}(xy) = x^{-1}0 = 0$  and hence  $(x^{-1}x)y = 0$ , which implies  $y = 0$ .

**Exercise 21** Calculate  $123456^{78} \pmod{5}$ .

**Solution:** 1 (as  $123456 \pmod{5} = 1$ )



**Exercise 22** Let  $n$  be a positive integer. How many different local maximum points has the real function

$$f(x) = \prod_{k=0}^{2n+1} (x - k)?$$

**Solution:**  $n$

**Exercise 23** Let  $G$  be a simple undirected graph with  $n$ ,  $n \geq 1$ , vertices and  $m$  edges such that  $m \geq n$ . Show that  $G$  has a cycle.

**Solution:** We prove the statement by contradiction. Assume  $G$  has no cycle. Then there is at most one path between any two vertices of  $G$ . This implies that any edge of  $G$  is a bridge, which shows that  $G$  is a forest. We can assume that  $G$  is connected, which means it is a tree. Otherwise we insert additional edges until it becomes a tree. However, a tree of order  $n$  has exactly  $n - 1$  edges, which contradicts the initial assumption  $m \geq n$ . That a tree of order  $n$  has indeed exactly  $n - 1$  edges can be shown by induction on  $n$ .

**Exercise 24** Find the value of the following integral:

$$\int_0^1 \int_0^x \int_0^y yz \, dz \, dy \, dx$$

**Solution:**

$$\begin{aligned} \int_0^1 \int_0^x \int_0^y yz \, dz \, dy \, dx &= \int_0^1 \int_0^x \frac{yz^2}{2} \Big|_0^y \, dy \, dx \\ &= \frac{1}{2} \int_0^1 \int_0^x y^3 \, dy \, dx \\ &= \frac{1}{2} \int_0^1 \frac{y^4}{4} \Big|_0^x \, dx \\ &= \frac{1}{8} \int_0^1 x^4 \, dx = \frac{1}{40}. \end{aligned}$$

**Exercise 25** Solve the differential equation

$$\frac{dy}{dt} = y + t$$

for the initial value  $y(0) = 1$ .

**Solution:**  $y(t) = 2e^t - t - 1$  You can solve first the homogeneous differential equation  $\frac{dy}{dt} = y$ , which gives  $y(t) = Ae^t$ . Now you know that the general solution should look like  $y(t) = Ae^t + Bt + C$ . Substituting this general solution for  $y$  in the given differential equation provides  $B = C = -1$ . Now you can use the initial value to find  $A = 2$ .