## Lösungen zu den Übungsaufgaben Differentialrechnung

1. a) Parallelebene zur x-y-Ebene, $D_f = \mathbb{R}^2, \quad W_f = \{c\}$
   b) $D_f = \mathbb{R}^2, \quad W_f = \mathbb{R}$, Ebene, Achsenabschnittsform verwenden
   c) $D_f = \mathbb{R}^2, \quad W_f = [-1, 1]$, "Wellpappe"
   d) $D_f = \{(x, y)^T \in \mathbb{R}^2 | y \in \mathbb{R}, -1 \leq x \leq 1\}, \quad W_f = [0, 1]$, "Tunnel" mit halbkreisförmigem Querschnitt
   e) $D_f = \mathbb{R}^2, \quad W_f = [0, \infty)$, Kegel mit Spitze in $(0, 0)^T$
   f) $D_f = \{(x, y)^T \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}, \quad W_f = [0, 1]$, obere Hälfte der Einheitskugel
   g) $D_f = \mathbb{R}^2 \setminus \{(0, 0)^T\}, \quad W_f = (0, \infty)$, Hyperboloid ("Kühlturm")

2. a) $-13.11$
   b) 1
   c) Grenzwert existiert nicht

3. a) überall stetig
   b) in $D_f$ stetig
   c) unstetig für $x = y$
   d) unstetig auf dem Rand des Einheitskreises
   e) unstetig für $xy = 0$
   f) unstetig für $x = \frac{k\pi}{2}, \quad k \in \mathbb{Z}$
   g,h) unstetig entlang der x- und der y-Achse

4. a) $f_x = -\frac{y^2}{(x-y)^2}, \qquad f_y = \frac{x^2}{(x-y)^2}$
   b) $f_x = (1 - xy)e^{-yx}, \quad f_y = -x^2 e^{-xy}$
   c) $f_s = e^{\sin(st)} \cos(st) \cdot t, \quad f_t = e^{\sin(st)} \cos(st) \cdot s$
   d) $f_x = yz - \frac{y-z}{x^2}, \quad f_y = xz + \frac{1}{x}, \quad f_z = xy - \frac{1}{x}$
   e) $f_u = \frac{-u^2x+xv^2+xw^2}{(u^2+v^2+w^2)^2}, \quad f_v = \frac{-2uvx}{(u^2+v^2+w^2)^2}, \quad f_w = \frac{-2uwx}{(u^2+v^2+w^2)^2}$
   f) $f_u = \frac{u}{\sqrt{u^2+v^2}}, \quad f_v = \frac{v}{\sqrt{u^2+v^2}}$
   g) $f_u = (\frac{u}{v})^w \frac{w}{u}, \quad f_v = -(\frac{u}{v})^w \frac{w}{v}, \quad f_w = (\frac{u}{v})^w \ln \frac{u}{v}$

5. a) $f_x = x, \qquad f_y = y, \qquad f_{xy} = f_{yx} = 0, \quad fxx = f_{yy} = 1$
   b) $f_x = y^x \ln y, \quad f_y = xy^{x-1}, \quad f_{xy} = f_{yx} = y^{x-1} + xy^{x-1} \ln y, \quad f_{xx} = y^x (\ln y)^2,$
   $f_{yy} = x(x - 1)y^{x-2}$
   c) $f_x = 2x + ye^x + 2xe^y - 3 \ln y, \quad f_y = e^x + x^2 e^y - \frac{3x}{y}, \quad f_{xy} = f_{yx} = e^x + 2xe^y - \frac{3}{y},$
   $f_{xx} = 2 + ye^x + 2e^y \quad f_{yy} = x^2 e^y + \frac{3x}{y^2}$
   d) $f_x = -\frac{y}{x^2+y^2}, \qquad f_y = \frac{x}{x^2+y^2}, \quad f_{xy} = f_{yx} = \frac{-x^2+y^2}{(x^2+y^2)^2}, \quad f_{xx} = \frac{2xy}{(x^2+y^2)^2}, \quad f_{yy} = -\frac{2xy}{(x^2+y^2)^2}$

6. a) $\frac{df}{dt} = 4t^3 + 3t^2 + 2t$
   b) $\frac{df}{dt} = -\frac{1}{e^t}$
   c) $\frac{df}{dx} = e^y + xe^y \frac{dy}{dx}$
   d) $\frac{df}{dt} = 2B \cos(2t) + (A - C) \sin(2t)$
   e) $\frac{df}{du} = e^{u+v}(-v \sin(uv) + \cos(uv)); \quad \frac{df}{dv} = e^{u+v}(-u \sin(uv) + \cos(uv));$

7. a) $y' = \frac{1}{3y^2+3}$
   b) $y' = \frac{1}{1-0.5 \cos y}$
   c) $y' = -\sqrt[3]{\frac{y}{x}}$
   d) $y' = -\frac{6x^2+2y^2}{4xy-2}$

e) $y' = -\frac{1}{1-e\cos y}$

8. a) $df = -\frac{2y}{(x-y)^2}dx + \frac{2x}{(x-y)^2}dy$
   b) $du = -\frac{yz+x^2}{x^2z}dx + \frac{y^2-xz}{xy^2}dy + \frac{z^2+xy}{yz^2}dz$
   c) $dw = (5x^4 + 18x^2y - 4xyz)dx + (6x^3 - 2x^2z + 3z^2)dy + (-2x^2y + 6yz)dz$
   d) $du = e^x \ln y dx + (\frac{e^x}{y} - z^2 \sin y)dy + 2z \cos y dz$
   e) $df = e^{\alpha^2-\beta}[2(\beta + \gamma^2)\alpha d\alpha + 2\gamma d\gamma + (1 - \beta - \gamma^2)d\beta$

9. a) ja  b) nein  c) ja, außer bei $y = 0$

10. a) $3ex - ey - z = 3e$,  b) $z + 4x - 6y = 5$

11. $|\triangle\rho| \lesssim 0.001405\frac{g}{cm^3}$,  $\frac{|\triangle\rho|}{|\rho|} \lesssim 0.894\%$

12. $|\triangle J_x| \lesssim 0.000362kgm^2$,  $\frac{|\triangle J_x|}{|J_x|} \lesssim 3.5\%$

13. $|\triangle F_{KN}| \lesssim 18505,3N$;  $\frac{|\triangle F_{KN}|}{|F_{KN}|} \lesssim 5.1\%$

14. $|\triangle\sigma| \lesssim 75 \cdot 10^6 N/m^2$,  $\frac{|\triangle\sigma|}{|\sigma|} \lesssim 5\%$

15. $\frac{|\triangle I|}{|I|} \lesssim 2.1\%$

16. $\frac{|\triangle F_{KN}|}{|F_{KN}|} \lesssim 5.16\%$

17. $dV = -30\pi cm^3 \simeq -94.25cm^3$  (Rechnung mit dem Differential ohne Beträge)

18. $|\triangle\alpha| \lesssim 0.0005$,  $\frac{|\triangle\alpha|}{|\alpha|} \lesssim 0.00094 = 0.094\%$

19. $\frac{df}{d\underline{s}}|_{\underline{x}=\underline{x}_0} = gradf|_{\underline{x}_0} \cdot \underline{s} = \tan\phi_{\underline{s}}$

20. a) $f_x|_P = 3$
    b) $f_y|_P = 4$
    c) $\frac{df}{d\underline{l}}|_P = \frac{7}{\sqrt{2}}$
    d) $\frac{df}{d\underline{l}}|_P = \frac{23}{\sqrt{26}}$
    e) $\frac{df}{d\underline{l}}|_P = 5$
    f) in Richtung des Gradienten

21. a) $\phi_1 = 89,18^o$
    b) $\phi_2 = -88.45^o$
    c) $\phi_3 = -60,79^o$
    d)*) $l_1 = (-1, 0.75)^T$,  $l_2 = (-1, -2.138)^T$

22. a) Minimum in $(2, -1, 1)^T$
    b) Sattelpunkt
    c) Maximum in $(0,0,1)^T$
    d) Minimum in $(-2, -6, -44)^T$
    e) Sattelpunkt
    f) Minimum in $(1, 0.5, 0)^T$
    g) Minimum in $(-2, 0, -0.735)^T$

h) Maxima in $(2l\pi, 0, 2)$ mit $l \in \mathbb{Z}$
i)*) Minima in $(\pm\sqrt{\ln 2}, 0, 0.846)^T$
k) Minimum in $(\frac{10}{3}, -5, -\frac{200}{9})^T$ und Maximum in $(-\frac{10}{3}, -5, \frac{3800}{9})^T$

## Lösungen zu den Übungsaufgaben Integralrechnung

1. a) $NB_X = NB_Y = \{\binom{x}{y} \in \mathbb{R}^2 \mid a \le x \le b; c \le y \le d\}$

   b) $NB_X = \{\binom{x}{y} \in \mathbb{R}^2 \mid 0 \le x \le a; x \le y \le a\}; \quad NB_Y = \{\binom{x}{y} \in \mathbb{R}^2 \mid 0 \le y \le a; 0 \le x \le y\}$

   c) $NB_X = \{\binom{x}{y} \in \mathbb{R}^2 \mid -1 \le x \le 1; 0 \le y \le -x^2 + 1\};$

   $NB_Y = \{\binom{x}{y} \in \mathbb{R}^2 \mid 0 \le y \le 1; -\sqrt{1-y} \le x \le \sqrt{1-y}\}$

   d) $NB_X = \{\binom{x}{y} \in \mathbb{R}^2 \mid 0 \le x \le 1; -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}\};$

   $NB_Y = \{\binom{x}{y} \in \mathbb{R}^2 \mid -1 \le y \le 1; 0 \le x \le \sqrt{1-y^2}\}$

   e) $NB_X = \{\binom{x}{y} \in \mathbb{R}^2 \mid -1 \le x \le 0; -x-1 \le y \le x+1\} \cup$
   $\cup \{\binom{x}{y} \in \mathbb{R}^2 \mid 0 \le x \le 1; x-1 \le y \le -x+1\};$

   $NB_Y = \{\binom{x}{y} \in \mathbb{R}^2 \mid -1 \le y \le 0; -y-1 \le x \le y+1\} \cup$
   $\cup \{\binom{x}{y} \in \mathbb{R}^2 \mid 0 \le y \le 1; y-1 \le x \le -y+1\}$

   f) $NB_X = \{\binom{x}{y} \in \mathbb{R}^2 \mid -1 \le x \le 0; -\sqrt{x+1} \le y \le \sqrt{x+1}\} \cup$
   $\cup \{\binom{x}{y} \in \mathbb{R}^2 \mid 0 \le x \le 3; x-1 \le y \le \sqrt{x+1}\};$

   $NB_Y = \{\binom{x}{y} \in \mathbb{R}^2 \mid -1 \le y \le 2; y^2 - 1 \le x \le y+1\}$

2. a) $I = \int_0^2 \int_{y/2}^{y} f(x,y)dxdy + \int_2^4 \int_{y/2}^{2} f(x,y)dxdy$

   b) $I = \int_0^1 \int_{\sqrt{y}}^{\sqrt[3]{y}} f(x,y)dxdy$

   c) $I = \int_0^4 \int_0^x f(x,y)dydx + \int_4^6 \int_0^4 f(x,y)dydx + \int_6^{10} \int_0^{10-x} f(x,y)dydx$

   d) $I = \int_{-1}^0 \int_0^{\sqrt{1-x^2}} f(x,y)dydx + \int_0^1 \int_0^{1-x} f(x,y)dydx$

3. a) $(e-1)^2$ ; b) $\frac{13}{3}\ln 2 - \frac{59}{36} \approx 1.365$; c) -2

4. a) $\frac{1}{24} \approx 0.0417$ b) $\frac{101}{60} \approx 1.683$

5. a) $V = \int_0^1 \int_0^{1-x} (1 + x + y - 0)\, dydx = \frac{5}{6}$

   b) $V = \int_0^1 \int_0^{1-x} \int_0^{1+x+y} 1dzdydx$

   c) $A = \iint\limits_A 1dA = \int\limits_a^b \int\limits_{y_1=y_1(x)}^{y_2=y_2(x)} 1dydx = \int\limits_a^b (y_2(x) - y_1(x))\, dx$

   $V = \iiint\limits_V 1dV = \iint\limits_A \int\limits_{z_1(x)}^{z_2(x)} 1dAdz = \iint\limits_A (z_2(x) - z_1(x))\, dA = \int\limits_a^b \int\limits_{y_1=y_1(x)}^{y_2=y_2(x)} (z_2(x) - z_1(x))\, dydx$

   d) s. Vorlesung!

6. a) $\frac{e-1}{2e} \approx 0.316$; b) $\frac{1}{2}\left(e^4 - e^2 - 2e\right) \approx 20.89$ c) $2\pi + 2 \approx 8.28$

7. a) $A = 2$; $x_s = \frac{4}{3}$; $y_s = 2$

   b) $A = 2 + \frac{\pi}{2} \approx 3.57$; $x_s \approx 0.1867$; $y_s = -0.1867$

8. $m = \frac{36}{35} \approx 1.0286$; $x_s = \frac{41}{27} \approx 1.52$; $y_s = 0$

9. $I = \frac{1}{6}$

10. $V = \frac{128}{3} \approx 42.67$

11. $V = 108$; $x_s = \frac{27}{16} \approx 1.69$

12. $I = \frac{1}{2}\left(\frac{5}{8} - \ln 2\right) \approx 0.034$

13. $V = \frac{28}{15} \approx 1.87$

14. $I_1 = \pi$; $I_2 = 2\pi$

15. $r = a$; $A = \frac{1}{4}\pi a^2$; $x_s = y_s = \frac{4}{3}\frac{a}{\pi}$

16. $x_s = y_s = 0$; $V \approx 17.96$; $z_s \approx 2.8$

17. $V = 12\pi$

18. $m = \frac{1}{2}\left(3e^4 + 1\right)\pi$

19. $x_s = y_s = 0$; $V = \frac{128}{3}\pi$; $z_s = 6$

20. $J_x = \frac{\pi}{12}$

21. $Q = \frac{5}{4}\rho_0\pi$

## Lösungen zu den Übungsaufgaben Kurvenintegrale

1. a) $s = \frac{3}{4} - \frac{1}{2}\ln 2 \approx 1.096$; b) $s = \sqrt{3}\left(e^b - 1\right)$

2. $I = 1$

3. $I = 2a^2$

4. a) $K = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} | y = x^2; \quad 1 \leq x \leq 2 \right\}$ *bzw.*
   $K = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} | t = x;\; y = t^2; \quad 1 \leq t \leq 2 \right\}$
   b) $K = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} | y = 3x - 2; \quad 1 \leq x \leq 2 \right\}$ *bzw.*
   $K = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} | t = x;\; y = 3t - 2; \quad 1 \leq t \leq 2 \right\}$
5. a) $I = 6 + \frac{7}{12} \approx 6.583$; b) $I = 7$
6. $I = 16$
7. a) $I = \frac{2}{3}$; b) $I = \frac{4}{3}$
8. a) $-\pi$ b) $\pi$ c) 0 d) $\frac{8}{3}$
9. s. Vorlesung!
10. a) wegunabhängig, $I = 3$; b) nicht wegunabhängig, $I = -4 + 2e$;
    c) wegunabhängig, $I = -24$
11. nicht wegunabhängig, $J = I$
12. wegunabhängig, $I = 0$
13. $I = 0 \Longleftrightarrow a = b = c$

**Lösungen zu den Übungsaufgaben Vektoranalysis**

1. a) s. Vorlesung!, grad $\overrightarrow{r} = \frac{\overrightarrow{r}}{r}$
2. a) $\text{grad}\, r^n = nr^{n-2}\overrightarrow{r}$; b) $\text{grad} \ln r = \frac{\overrightarrow{r}}{r^2}$ c) $\text{grad}\, e^r = e^r \frac{\overrightarrow{r}}{r}$
   d) $\text{grad} \sin r = \cos r \frac{\overrightarrow{r}}{r}$
3. a) $\text{grad}\, U|_P = \begin{pmatrix} -12 \\ -9 \\ -16 \end{pmatrix}$; Richtung des größten Wachstums
   b) $\text{grad}\, V|_P = \dfrac{1}{2\sqrt{3}} \begin{pmatrix} 0 \\ 6 \\ -1 \end{pmatrix}$
4. $\text{grad}\, U|_P = \begin{pmatrix} -4 \\ -4 \\ 12 \end{pmatrix}$
5. s. Vorlesung!, div $\overrightarrow{v} = 3$; $rot\ \overrightarrow{v} = (0;0;0)^T$
6. a)div $\overrightarrow{r} = 2(x + y + z)$; $rot\ \overrightarrow{r} = (-1;-1;-1)^T$
   b) div $\overrightarrow{v} = ye^{xy} + xz + x^2ye^z$; $rot\ \overrightarrow{v} = (x^2e^z - xy; -2xye^z; yz - xe^{xy})^T$
   c) div $\overrightarrow{v} = (3 + n)r^n$; $rot\ \overrightarrow{v} = (0;0;0)^T$
   d) div $\overrightarrow{v} = 3 \ln r + 1$; $rot\ \overrightarrow{v} = (0;0;0)^T$

7. $a = -2$

8. a) nicht konservativ
   b) konservativ, $U = xy + xz + zy + C$
   c) konservativ, $U = \frac{1}{2}(x^2 + y^2) + C$

9. * nutze Umformungsregeln für Rotation und Gradient

10. $1 - e$

11. 2

12. $\frac{1}{4}R^4 h\pi$