Basics of Functional Analysis

1 Metric Spaces

Definition 1 *A* ***metric space*** $\mathbb{X}$ *is defined to be a nonempty set* $\mathbb{X}$ *together with a real function* d, $\mathbb{X} \times \mathbb{X} \to \mathbb{R}$, *satisfying 3 conditions:*

1. $d(\mathbf{x},\mathbf{y}) \geq 0, \quad \wedge \quad d(\mathbf{x},\mathbf{y}) = 0 \iff \mathbf{x} = \mathbf{y}$
   (nonnegativity, nondegeneracy)
2. $d(\mathbf{x},\mathbf{y}) = d(\mathbf{y},\mathbf{x}) \quad \forall \mathbf{x},\mathbf{y} \in \mathbb{X}$ (symmetry)
3. $d(\mathbf{x},\mathbf{y}) \leq d(\mathbf{x},\mathbf{z}) + d(\mathbf{z},\mathbf{y}) \quad \forall \mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{X}$ (triangle inequality)

$d(\mathbf{x},\mathbf{y})$ is called the distance function or the **Metric in** $\mathbb{X}$.

Definition 2 *Let* $(\mathbb{X}, d)$ *be a metric space,* $\mathbf{x}_0 \in \mathbb{X}$*;*
The set $K_\varepsilon(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{X} \mid d(\mathbf{x},\mathbf{x}_0) < \varepsilon\}$ *is called an* ***open ball*** *centered at* $\mathbf{x}_0$ *with the radius* ε*, or is called the* $\varepsilon-$ ***neighbourhood of*** $\mathbf{x}_0$.

Definition 3 *The proper subset* $A \subset \mathbb{X}$ *is called* ***open****,*
if $\forall \mathbf{x} \in A \quad \exists r > 0 \mid K_r(\mathbf{x}) \subseteq A$.

Definition 4 *The proper subset* $U \subset \mathbb{X}$ *is called the* ***neighbourhood of*** $\mathbf{x}_0$*,*
if it contains an $\varepsilon-$*neighbourhood of* $\mathbf{x}_0$.

Definition 5 $\mathbf{x}_0$ *is an interior point of* $A \iff \exists \varepsilon > 0 \mid K_\varepsilon(\mathbf{x}_0) \subset A$

Definition 6 *The proper subset* $A \subset \mathbb{X}$ *is called* ***bounded****, if* A *is completely contained within an open ball* $K_r(\mathbf{y})$, $\mathbf{y} \in \mathbb{X}$*,* $0 < r < \infty$.

Definition 7 *The point* $\mathbf{x}_0 \in \mathbb{X}$ *is a limit point of a set* $A \subset \mathbb{X}$ *, if every open ball centered at* $\mathbf{x}_0$ *contains a point* $\mathbf{x} \in A$*;* $\quad \mathbf{x} \neq \mathbf{x}_0$*. The set of all limit points of* A *is called the* ***derivated set*** A^+.

Definition 8 *The set* $\overline{A} = A \cup A^+$ *is called the* ***closure*** *or the* ***closed cover*** *of* A.

Definition 9 *The subset* $A \subset X$ *is called* ***closed****, if* $A^+ \subseteq A$.

Definition 10 *The set* B *is called* ***dense in*** A*, if* $B \subset A \quad \wedge \quad \overline{B} = A$.

Definition 11 *A **sequence** $\{\mathbf{x}_n\}_{n=1}^{\infty} \subset \mathbb{X}$ **is called convergent**, if there exists an element $\mathbf{x}_0 \in \mathbb{X}$ fulfilling the condition $\lim_{n\to\infty} d(\mathbf{x}_n, \mathbf{x}_0) = 0$. $\mathbf{x}_0$ is called the **limit** of the sequence.*

Definition 12 *Cauchy sequence*
*A sequence $\{\mathbf{x}_n\}_{n=1}^{\infty} \subset \mathbb{X}$ is said to be **Cauchy**, if given any $\varepsilon > 0$ there exists an integer $n_0(\varepsilon)$ such that $d(\mathbf{x}_n, \mathbf{x}_m) < \varepsilon$ whenever $n, m > n_0(\varepsilon)$,*
i.e. $\lim_{n,m\to\infty} d(\mathbf{x}_n, \mathbf{x}_m) = 0$.

Definition 13 *A metric space $\mathbb{X}$ is **complete**, if every Cauchy sequence in $\mathbb{X}$ converges to a point of $\mathbb{X}$.*

Definition 14 *Let $(\mathbb{X}, d)$ be a metric space. Subset $A \subset \mathbb{X}$ is called **sequentially compact**, if every sequence $\{\mathbf{x}_n\}_{n=1}^{\infty} \subset A$ contains a convergent subsequence $\{\widetilde{\mathbf{x}}_n\}_{n=1}^{\infty} \subset A$ with $lim_{n\to\infty}\mathbf{x}_n^{'} = \mathbf{x} \in \mathbb{X}$.*

Definition 15 *The subset $A \subset \mathbb{X}$ is called c**ompact**, if every sequence $\{\mathbf{x}_n\}_{n=1}^{\infty} \subset A$ contains a convergent subsequence $\{\widetilde{\mathbf{x}}_n\}_{n=1}^{\infty} \subset A$*
with $lim_{n\to\infty}\widetilde{\mathbf{x}}_n = \mathbf{x} \in A$.

Notation 16 *$A \subset \mathbb{X}$ is compact $\iff$ $A \subset \mathbb{X}$ is sequentially compact and closed.*

1.1 Operators

Definition 17 *A unique mapping from A to B, $T : A \to B$ is called an operator: $T\mathbf{x} = \mathbf{y}$*

Definition 18 *The range of T is the set*
$T(A) = \{\mathbf{y} \in B \mid \exists \mathbf{x} \in A \text{ with } T(\mathbf{x}) = \mathbf{y}\}$.

Definition 19 *The operator $T : A \to B$ is called*

- **surjective (onto)** $\iff$ $T(A) = B$
- **injective** (one-to-one) $\iff$ $T(x) = T(y) \quad \curvearrowright \quad x = y$
- **bijective** $\iff$ T is surjective and injective.

Definition 20 *If the operator $T : A \to B$ is bijective, then there exists the **inverse operator** $T^{-1} : B \to A$, which is defined by $T^{-1}\mathbf{y} = \mathbf{x} \iff T\mathbf{x} = \mathbf{y}$.*

Definition 21 *The* ***operator*** $T : A \to B$ ***is called continuous at*** $\mathbf{x}_0 \in A$, *if* $\forall \varepsilon > 0 \quad \exists \delta = \delta(x_0, \varepsilon) \quad | \quad d_y(T\mathbf{x}, T\mathbf{x}_0) < \varepsilon \quad \forall \mathbf{x} \in A$ *with* $d_x(\mathbf{x}, \mathbf{x}_0) < \delta$. *If* T *is continuous at every point* $\mathbf{x}_0 \in A$, *then* T *is called* ***continuous on*** A. *In Addition, if* $\delta(x_0, \varepsilon)$ *is independent of* x_0 *for all* ε *then* T *is called* ***uniformly continuous on*** A.

Definition 22 *A bijective continuous mapping* $T : A \to B$ *, with a continuous inverse mapping is called a* **homeomorphism***. Two set are called homeomorph* $\Leftrightarrow \exists$ *homeomorphism* $T : A \to B$.

Definition 23 *Let* A *be closed. The mapping* $T : A \to A$ *is called a contraction mapping, if there exists a number* $0 < q < 1$ *such that*
$d(T\mathbf{x}, T\mathbf{y}) \leq q \cdot d(\mathbf{x}, \mathbf{y}) \ \forall \ \mathbf{x}, \mathbf{y} \in A$.

Theorem 24 ***BANACH Fixed Point Theorem (FPT)***
Let A *be a closed subset of the complete metric space* $(\mathbb{X}, d)$ *with a contraction mapping* $T : A \to A$. *Then* T *admits a unique fixed-point* $\mathbf{x}^* \in A$, *i.e.* $\mathbf{x}^*$ *is the solution of the fixed point equation* $\mathbf{x} = T\mathbf{x}$.
Then the iterative sequence $\{\mathbf{x}_n\}_{n=0}^{\infty}$ *which starts with an arbitrary element* $\mathbf{x}_0 \in A$, *defined by* $x_{n+1} = T\mathbf{x}_n$, *tends to* $\mathbf{x}^*$ *as* n *tends to infinity. The following inequalities are true and describe the speed of convergence:*

- a priori : $d(\mathbf{x}_n, \mathbf{x}^*) \leq \frac{q^n}{1-q} d(\mathbf{x}_0, \mathbf{x}_1)$
- a posteriori : $d(\mathbf{x}_n, \mathbf{x}^*) \leq \frac{q}{1-q} d(\mathbf{x}_n, \mathbf{x}_{n-1})$.

2 Linear Spaces

Definition 25 *A vector space over a field* $\mathbb{K}$ *(* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{K} = \mathbb{C}$ *) is a nonempty set* $\mathbb{X}$ *together with two binary operations that satisfy the eight axioms listed below. (Elements of* $\mathbb{X}$ *are called vectors. Elements of* $\mathbb{K}$ *are called scalars.)*

(A) The first operation, **addition**, takes any two elements $\mathbf{x} \in \mathbb{X}$, $\mathbf{y} \in \mathbb{X}$ and assigns to them a third, unique element which is commonly written as $\mathbf{x} + \mathbf{y} \in \mathbb{X}$ and called the sum of these two elements.

(M) The second operation takes any scalar $\lambda \in \mathbb{K}$ and any vector $\mathbf{x} \in \mathbb{X}$ and gives another unique element $\lambda\mathbf{x} \in \mathbb{X}$. The multiplication is called the **scalar multiplication** of $\mathbf{x}$ by λ.

To qualify as a vector space, the set $\mathbb{X}$ and the operations of addition and scalar multiplication must adhere to a number of requirements called the **Axioms of the linear space**.
Let $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ be arbitrary vectors in $\mathbb{X}$, and λ and μ be scalars in $\mathbb{K}$.

(A1) $\mathbf{x}+(\mathbf{y}+\mathbf{z})=(\mathbf{x}+\mathbf{y})+\mathbf{z}$ (**Associativity of addition**)

(A2) $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$ (**Commutativity of addition**)

(A3) There exists a unique element $\mathbb{O}\in\mathbb{X}$, such that $\mathbf{x}+\mathbb{O}=\mathbf{x} \quad \forall\mathbf{x}\in\mathbb{X}$
(**zero/identity element of addition**)

(A4) For every $\mathbf{x}\in\mathbb{X}$, there exists a unique element $(-\mathbf{x})\in\mathbb{X}$ such that $\mathbf{x}+(-\mathbf{x})=\mathbb{O}$
(**Inverse element of addition**)

(M1) $(\lambda+\mu)\mathbf{x}=\lambda\mathbf{x}+\mu\mathbf{x}$ (**Distributivity of scalar multiplication with respect to field addition**)

(M2) $\lambda(\mathbf{x}+\mathbf{y})=\lambda\mathbf{x}+\lambda\mathbf{y}$ (**Distributivity of scalar multiplication with respect to vector addition**)

(M3) $(\lambda\mu)\mathbf{x}=\lambda(\mu\mathbf{x})$ (**Compatibility of scalar multiplication with field multiplication**)

(M4) $1\mathbf{x}=\mathbf{x}$; ($1\in\mathbb{K}$, **Identity element of scalar multiplication**)

Notation 26 *If $\mathbb{K}=\mathbb{R}$ ($\mathbb{K}=\mathbb{C}$) then we get a real (complex) linear space.*

Definition 27 *Let $\mathbb{U}$ be a subset of $\mathbb{X}$. Then $\mathbb{U}$ is a subspace if and ony if it satisfies the following conditions:*
a) If $\mathbf{x}$ and $\mathbf{y}$ are elements of $\mathbb{X}$, then the sum $\mathbf{x}+\mathbf{y}$ is an element of $\mathbb{U}$
b) If $\mathbf{x}$ is an element of $\mathbb{U}$ and λ is a scalar from $\mathbb{K}$, then the scalar product $\lambda\mathbf{x}$ is an element of $\mathbb{U}$

Definition 28 *Let $\mathbb{U}$ be a subspace of $\mathbb{X}$ and $x_0\in\mathbb{X}$ then*

$$M=\{\mathbf{x}_0+\mathbf{y}\mid\mathbf{y}\in\mathbb{U}\}\equiv\mathbf{x}_0+\mathbb{U}$$

*is called **linear manifold** in $\mathbb{X}$.*

Definition 29 *Let A be a subset $A \subset \mathbb{X}$. The set of all finite linear combinations of elements of A*

$$spanA = \left\{ \sum_{k=1}^{m} \lambda_k \mathbf{x}_k \quad | \quad \mathbf{x}_k \in A, \quad \lambda_k \in \mathbb{K}, \quad m \in \mathbb{N} \right\}$$

is called the ***linear span of*** *A*

Definition 30 *Let $\mathbb{U}$ and $\mathbb{V}$ be subspaces of $\mathbb{X}$, then*

$$\mathbb{U} + \mathbb{V} = span(U \cup V)$$

is called the ***sum of*** *$\mathbb{U}$* ***and*** *$\mathbb{V}$. Additionally, if $\mathbb{U} \cap \mathbb{V} = \{\mathbb{O}\}$,*
then $\mathbb{U} + \mathbb{V}$ is called the ***direct sum*** *$\mathbb{U} \oplus \mathbb{V}$. Every $\mathbf{z} \in \mathbb{U} \oplus \mathbb{V}$ has a unique representation in the form $\mathbf{z} = \mathbf{x} + \mathbf{y}$ with $\mathbf{x} \in \mathbb{U}$ and $\mathbf{y} \in \mathbb{V}$.*

Definition 31 *If $\mathbb{X} = \mathbb{U} \oplus \mathbb{V}$, then the subspaces $\mathbb{U} \subset \mathbb{X}$ and $\mathbb{V} \subset \mathbb{X}$ are called* ***complementary.***

Definition 32 *The set $\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n\} \subset \mathbb{X}$ is called linearly independent, if*

$$\sum_{k=1}^{m} \lambda_k \mathbf{x}_k = \mathbf{0} \Longleftrightarrow \lambda_1 = \lambda_2 = ... = \lambda_n = 0$$

Definition 33 *The set $B \subset \mathbb{X}$ is called* ***linearly independent****, if every finite subset of B is linearly independent.*

Definition 34 *A linearly independent subset $B \subset \mathbb{X}$ with $\mathbb{X} = \overline{spanB}$ is called a* ***basis*** *in $\mathbb{X}$.*

Definition 35 *If there exists a basis of $\mathbb{X}$ with $|B| = n$, then every basis of $\mathbb{X}$ consists of n elements: $\dim \mathbb{X} = n$. If there is no finite n then $\mathbb{X}$ is called* ***infinitely dimensional****.*

Definition 36 *Let $\mathbb{X}$ and $\mathbb{Y}$ be linear spaces over $\mathbb{K}$. $\mathbb{X}$ and $\mathbb{Y}$ are said to be* ***linear isomorph****, if there exists a bijection $f \colon \mathbb{X} \to \mathbb{Y}$ with the property*

$$f(\alpha \mathbf{x} + \beta \mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}) \qquad \forall \mathbf{x}, \mathbf{y} \in \mathbb{X}; \quad \alpha, \beta \in \mathbb{K}$$

2.1 Normed Linear Spaces

Definition 37 *A (real)* ***normed linear space*** $(\mathbb{V}, \|.\|)$ *is a (real) linear space* $\mathbb{V}$ *over the field* $\mathbb{K}$ *together with a function* $\|.\|$ *,* $\mathbb{V} \to \mathbb{R}$*, called the norm, satisfying the following 3 conditions for any* $\mathbf{x}, \mathbf{y} \in \mathbb{V}$*:*

(I) $\|\mathbf{x}\| \geq 0 \quad \wedge \quad \|\mathbf{x}\| = 0 \quad \Longleftrightarrow \quad \mathbf{x} = \mathbf{0}$
(nonnegativity and nondegeneracy)

(II) $\|\alpha\mathbf{x}\| = |\alpha| \, \|\mathbf{x}\|, \quad \alpha \in \mathbb{K}$ (multiplicativity)

(III) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (triangle inequality)

$\|\mathbf{x}\|$ *is called the norm of the element* $\mathbf{x}$.

Notation 38 *In any linear normed space* $\mathbb{V}$ *you can introduce the c****anonical or induced metric by*** $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{V}$*. Therefore every linear normed space is a metric space.*

Definition 39 *A linear normed vector space* $\mathbb{V}$ *over the field* $\mathbb{K}$ *which is complete with respect to the metric* $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$*, induced by the norm, is called a* ***BANACH space****.*

Definition 40 *The set* $K_r(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{B} \mid \|\mathbf{x} - \mathbf{x}_0\| < r\}$ *is called an* ***open ball centered at*** $\mathrm{x}_0 \in B$ ***with the radius*** r*.*

Convergence in the BANACH space $\mathbb{B}$*:*

- Let $\{\mathbf{x}_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{B}$.
  $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}_0 \quad \Longleftrightarrow \quad \forall \varepsilon > 0 \; \exists n_0(\varepsilon) \mid \|\mathbf{x}_n - \mathbf{x}_0\| < \varepsilon \quad \forall n \geq n_0$
- $\{\mathbf{x}_n\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{B}$, if
  $\forall \varepsilon > 0 \; \exists n_0(\varepsilon) \mid \|\mathbf{x}_n - \mathbf{x}_m\| < \varepsilon \quad \forall n, m \geq n_0$
- Because of the completeness of the BANACH space, every Cauchy sequence tends to a limit in the BANACH space.
- Convergence in normed spaces is called **norm convergence**.
- The set of all norm convergent sequences is linear:
  $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x} \wedge \lim_{n\to\infty} \mathbf{y}_n = \mathbf{y}$ implies $\lim_{n\to\infty}(\mathbf{x}_n + \mathbf{y}_n) = \mathbf{x} + \mathbf{y}$.
  $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x} \wedge \lim_{n\to\infty} \alpha_n = \alpha$ implies $\lim_{n\to\infty} \alpha_n \mathbf{x}_n = \alpha\mathbf{x}$.
  $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}$ implies $\lim_{n\to\infty} \|\mathbf{x}_n\| = \|\mathbf{x}\|$.

Definition 41 *In a normed space* $\mathbb{V}$ *the norms* $\|.\|_1$ *and* $\|.\|_2$ *are said to be equivalent if* $\exists m, M \in \mathbb{R},\ m > 0, M > 0 \mid m \|\mathbf{x}\|_1 \leq \|\mathbf{x}\|_2 \leq M \|\mathbf{x}\|_1\ \forall \mathbf{x} \in \mathbb{V}$.

Definition 42 ***Series in Normed Spaces:***

Let $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n, ...$ *be elements of a linear normed space* $\mathbb{V}, \mathbf{s}_n = \sum_{k=1}^{n} \mathbf{x}_k$.

By definition the series $\sum_{k=1}^{\infty} \mathbf{x}_k$ *converges to a limit* $\mathbf{s} \in \mathbb{V}$ *if and only if the associated sequence of partial sums* $\{\mathbf{s}_n\}$ *converges to* $\mathbf{s}$ *i.e..*

$$\mathbf{s} = \sum_{k=1}^{\infty} \mathbf{x}_k \Longleftrightarrow \lim_{n \to \infty} \mathbf{s}_n = \mathbf{s}$$

$\mathbf{s} = \sum_{k=1}^{\infty} \mathbf{x}_k$ is called the s**um of the series** in $\mathbb{V}$.

Definition 43 *The series* $\sum_{k=1}^{\infty} \mathbf{x}_k$ *is called* ***absolutely convergent****, if the number series* $\sum_{k=1}^{\infty} \|\mathbf{x}_k\|$ *is convergent.*

2.2 Linear Operators

Definition 44 *Let* $\mathbb{X}, \mathbb{Y}$ *be linear normed spaces over the (same) field* $\mathbb{K}$**.** *A mapping* $\mathbf{A} : \mathbb{X} \to \mathbb{Y}$ *is called a* ***linear operator****, if:*

$$\begin{aligned} \mathbf{A}(\mathbf{u} + \mathbf{v}) &= \mathbf{A}\mathbf{u} + \mathbf{A}\mathbf{v} & \forall \mathbf{u}, \mathbf{v} \in \mathbb{X} \\ \mathbf{A}(\alpha \mathbf{u}) &= \alpha A \mathbf{u} & \forall \alpha \in \mathbb{K} \wedge \forall \mathbf{u} \in \mathbb{X}. \end{aligned}$$

Image space of A: $R(\mathbf{A}) = \{\mathbf{y} \in \mathbb{Y} \mid \mathbf{y} = \mathbf{A}\mathbf{x};\ \mathbf{x} \in \mathbb{X}\}$
Null space (kernel) of A: $N(\mathbf{A}) = \{\mathbf{x} \in \mathbb{X} \mid \mathbf{A}\mathbf{x} = \mathbf{0}\}$

Definition 45 *the linear operator* $\mathbf{A} : \mathbb{X} \to \mathbb{Y}$ *is called* ***bounded*** *if there exists some finite positive constant* $C \in \mathbb{R}$ *such that* $\|\mathbf{A}\mathbf{x}\| \leq C \|\mathbf{x}\| \quad \forall \mathbf{x} \in \mathbb{X}$.

Definition 46 *The number*

$$\begin{aligned} \|\mathbf{A}\| &= \inf \{C \in \mathbb{R} \,|\, \|\mathbf{A}\mathbf{u}\| \leq C \|\mathbf{u}\|, \forall \mathbf{u} \in \mathbb{X}\} \\ &= \sup_{\|\mathbf{u}\|=1} \|\mathbf{A}\mathbf{u}\| \end{aligned}$$

is called the ***norm of the operator.***

Theorem 47 *The linear operator* $\mathbf{A} : \mathbb{X} \to \mathbb{Y}$ *is* ***continuous*** *if and only if it is bounded.*

Definition 48 *A linear continuous operator* $\mathbf{A} : \mathbb{X} \to \mathbb{Y}$ *is called an* ***isomorphism*** *if it is bijective and if* $\mathbf{A}^{-1}$ *is continuous. That means: An isomorphism is a linear homeomorphism. Moreover if* $\|\mathbf{Ax}\| = \|\mathbf{x}\| \ \forall \mathbf{x} \in \mathbb{X}$ *is satisfied then* $\mathbf{A}$ *is called* ***isometric.*** *Normed spaces which are connected by an (isometric) isomophism are called (isometricly) isomorph.*

Definition 49 ***The sum*** $\mathbf{T}+\mathbf{S}$ *of the linear operators* $\mathbf{T}$ *and* $\mathbf{S}$ *is defined by the equation* $(\mathbf{T}+\mathbf{S})\mathbf{x} = \mathbf{Tx} + \mathbf{Sx} \quad \forall \mathbf{x} \in \mathbb{X}$*, the* ***product of the operators*** $\mathbf{T}$ ***by*** $\lambda \in \mathbb{R}$ *or* $\lambda \in \mathbb{C}$ *is defined by* $(\lambda \mathbf{T})\mathbf{x} = \lambda(\mathbf{Tx}) \quad \forall \mathbf{x} \in \mathbb{X}$.

Definition 50 *The collection* $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ *of all linear bounded operators* $\mathbf{A} : \mathbb{X} \to \mathbb{Y}$ *with the sum and the product defined above is called the* ***space of the linear bounded operators*** $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ *or the* ***dual space*** $\mathbb{X}'$ *of* $\mathbb{X}$.

Definition 51 *The sequence* $\{\mathbf{A}_n\}_{n=1}^{\infty} \subset \mathbb{L}(\mathbb{X}, \mathbb{Y})$ *is called* ***norm convergent*** *(strongly convergent) with the limit* $\mathbf{A} \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ *if*
$\lim_{n\to\infty} \|\mathbf{A}_n\mathbf{x} - \mathbf{Ax}\| = 0 \quad \forall \mathbf{x} \in \mathbb{X}$ *is satisfied.*
We write: $\mathbf{A}_n \overset{n\to\infty}{\rightrightarrows} \mathbf{A}$ *or* $\lim_{n\to\infty} \mathbf{A}_n = \mathbf{A}$.

Definition 52 *Let* $\mathbb{X}$*,* $\mathbb{Y}$*,* $\mathbb{Z}$ *be linear spaces and let* $\mathbf{T} : \mathbb{X} \to \mathbb{Y}; \quad \mathbf{S} : \mathbb{Y} \to \mathbb{Z}$ *be linear operators. Then the product* $\mathbf{ST}$ *of the operators is defined by*
$(\mathbf{ST})\mathbf{x} = \mathbf{S}(\mathbf{Tx}) \quad \forall \mathbf{x} \in \mathbb{X}$.

Definition 53 *Let* $\mathbf{T} : \mathbb{X} \to \mathbb{Y}$ *be a linear operator. If there exists a linear operator* $\mathbf{S} : \mathbb{Y} \to \mathbb{X}$ *such that :*
$\mathbf{ST} = \mathbf{I}_x \quad \wedge \quad \mathbf{TS} = \mathbf{I}_y;$ *with* $\mathbf{I}_x, \mathbf{I}_y$ *identity maps from* $\mathbb{X}$ *to* $\mathbb{X}$ *or* $\mathbb{Y}$ *to* $\mathbb{Y}$ *then* $\mathbf{S}$ *is the* ***inverse Operator*** *to:* $\mathbf{T}\ \mathbf{S} = \mathbf{T}^{-1}$.
The collection of all invertible operators $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$*is called* $\mathbb{L}_{inv}(\mathbb{X}, \mathbb{Y})$.

3 HILBERT Spaces

Definition 54 *An inner product space* $(\mathbb{H}, (.,.))$ *or pre-HILBERT space is a linear space* $\mathbb{H}$ *over the field* $\mathbb{K}$ *together with a function* $(.,.) : \mathbb{H} \times \mathbb{H} \to \mathbb{K}$*, which satisfies the following conditions:*

1. $(\mathbf{x}, \mathbf{x}) \geq 0 \qquad \wedge \qquad (\mathbf{x}, \mathbf{x}) = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$ *(nonnegativity and nondegeneracy)*
2. $(\mathbf{x}, \mathbf{y}) = \overline{(\mathbf{y}, \mathbf{x})}$ *(Hermitian symmetry)*

3. $(\alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z}) = \alpha(\mathbf{x}, \mathbf{z}) + \beta(\mathbf{y}, \mathbf{z}) \quad \forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{H}; \alpha, \beta \in \mathbb{K}$ *(linearity in the first argument).*

Definition 55 *Let $\mathbb{H}$ be a linear space over the field $\mathbb{K}$. The function $(.,.) : \mathbb{H} \times \mathbb{H} \to \mathbb{K}$ with the properties 1, 2, 3 is called the* ***inner product.***

Theorem 56 *Every pre-HILBERT space is a normed space with the norm* $\|\mathbf{x}\| = \sqrt{(\mathbf{x}, \mathbf{x})} \ \ \forall \mathbf{x} \in \mathbb{H}$

Properties of the inner product:

1. $(\mathbf{u}, \alpha\mathbf{v}) = \overline{\alpha}(\mathbf{u}, \mathbf{v}) \qquad \forall \mathbf{u}, \mathbf{v} \in \mathbb{H}, \ \alpha \in \mathbb{K}$
2. $(\mathbf{u}, \mathbf{v} + \mathbf{w}) = (\mathbf{u}, \mathbf{v}) + (\mathbf{u}, \mathbf{w}) \qquad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{H}$
3. $|(\mathbf{u}, \mathbf{v})| \leq \|\mathbf{u}\| \cdot \|\mathbf{v}\| \qquad \forall \mathbf{u}, \mathbf{v} \in \mathbb{H}$ SCHWARZ's Inequality

Theorem 57 *The inner product in a pre-HILBERT space is continuous, i.e.* $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}$ *and* $\lim_{n\to\infty} \mathbf{y}_n = \mathbf{y}$ *imply* $\lim_{n\to\infty}(\mathbf{x}_n, \mathbf{y}_n) = (\mathbf{x}, \mathbf{y})$.

Definition 58 *A* ***HILBERT space*** *is a complete pre-HILBERT space with respect to the metric $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$, induced by the norm $\|\mathbf{x}\| = \sqrt{(\mathbf{x}, \mathbf{x})}$.*

Definition 59 *Let $\mathbb{H}$ be a HILBERT space. The two elements $\mathbf{u}, \mathbf{v} \in \mathbb{H}$ are* ***orthogonal*** $(\mathbf{u} \perp \mathbf{v})$, *if* $(\mathbf{u}, \mathbf{v}) = 0$.

Definition 60 *Let $\mathbb{H}$ be a HILBERT space. A system $\{\mathbf{e}_i\}_{i=1}^{\infty}$ is called an* ***orthonormal system*** *(ONS) if* $(\mathbf{e}_i, \mathbf{e}_k) = \delta_{ik} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$.

The ONS is called ***closed or complete,*** *if* $\overline{\underset{i}{span}\{\mathbf{e}_i\}} = \mathbb{H}$.

Conclusion 61 $\forall \ \mathbf{u} \in \mathbb{H} \ ,\forall \varepsilon > 0 \quad \exists u_i \in \mathbb{R} \quad \wedge \quad \exists n_0(\varepsilon) \quad |$

$$\left\| \mathbf{u} - \sum_{i=1}^{n} u_i \mathbf{e}_i \right\| < \varepsilon \quad \forall n > n_0$$

Definition 62 *A HILBERT space $\mathbb{H}$ is called* ***separable,*** *if $\mathbb{H}$ has a countable subset $M = \{\mathbf{u}_1, \mathbf{u}_2, ...\}$ which is dense in $\mathbb{H}$, i.e. $\overline{M} = \mathbb{H}$.*

Theorem 63 *In a separable HILBERT space $\mathbb{H}$, there exists at least one complete ONS (and the contrary is true, too).*

Theorem 64 *Let $\mathbb{H}$ be a separable HILBERT space with the ONS $\{\mathbf{e}_i\}_{i=1}^{\infty}$, $\mathbf{u} \in \mathbb{H}$, $\mathbf{s}_n = \sum_{i=1}^{n} \gamma_i \mathbf{e}_i$; $\gamma_i \in \mathbb{C}$. Then we get:*

- $\|\mathbf{u} - \mathbf{s}_n\|$ is minimal for $\gamma_i = u_i = (\mathbf{u}, \mathbf{e}_i) \quad \forall i$
- $\lim_{n\to\infty} \sum_{i=1}^{n} u_i \mathbf{e}_i = \mathbf{s} \in \mathbb{H}$
- The series $\sum_{i=1}^{\infty} |u_i|^2$ converges and BESSEL's Inequality $\sum_{i=1}^{\infty} |u_i|^2 \leq \|\mathbf{u}\|^2$ is satisfied.
- If the ONS $\{\mathbf{e}_i\}_{i=1}^{\infty}$ is closed, then $\mathbf{s} = \mathbf{u}$ and we get PARSEVAL's equation: $\sum_{i=1}^{\infty} |u_i|^2 = \|\mathbf{u}\|^2$

Theorem 65 ***RIESZ - FISCHER***
Let $\{\mathbf{e}_i\}_{i=1}^{\infty}$ be an ONS in the HILBERT space $\mathbb{H}$ and $\{\gamma_i\}_{i=1}^{\infty}$ be a number series with
$\sum_{i=1}^{\infty} |\gamma_i|^2 < \infty \quad \Longrightarrow \quad \exists \mathbf{v} \in \mathbb{H} \mid (\mathbf{v}, \mathbf{e}_i) = \gamma_i \quad \forall i \quad \wedge \quad \mathbf{v} = \sum_{i=1}^{\infty} \gamma_i \mathbf{e}_i$.
If $\{\mathbf{e}_i\}_{i=1}^{\infty}$ is closed in $\mathbb{H}$, then $\mathbf{v}$ is well defined.

Definition 66 *Let $\mathbb{H}$ be a HILBERT space with the ONS $\{\mathbf{e}_i\}_{i=1}^{\infty}$, $\mathbf{u} \in \mathbb{H}$. The series $\sum_{i=1}^{\infty} (\mathbf{u}, \mathbf{e}_i)\mathbf{e}_i$ is called the* ***FOURIER series*** *of $\mathbf{u}$ with respect to the ONS $\{\mathbf{e}_i\}_{i=1}^{\infty}$,*
the numbers $u_i = (\mathbf{u}, \mathbf{e}_i)$ are called ***FOURIER coefficients****.*

Definition 67 *The HILBERT spaces $\mathbb{H}_1$, $\mathbb{H}_2$ are called isometric, if there exists a unique mapping $f : \mathbb{H}_1 \to \mathbb{H}_2$ with:*

- $f(\alpha \mathbf{u} + \beta \mathbf{v}) = \alpha f(\mathbf{u}) + \beta f(\mathbf{v})$
- $(\mathbf{u}, \mathbf{v}) = (f(\mathbf{u}), f(\mathbf{v})) \quad \forall \mathbf{u}, \mathbf{v} \in \mathbb{H}_1, \forall \alpha, \beta \in \mathbb{C}$

Definition 68 *The proper subspaces $M_1 \subset \mathbb{H}$ and $M_2 \subset \mathbb{H}$ of a HILBERT space $\mathbb{H}$ are called* ***orthogonal*** *if and only if the inner product*
$(\mathbf{u}, \mathbf{v}) = 0 \quad \forall \mathbf{u} \in M_1, \forall \mathbf{v} \in M_2$.

Definition 69 *For any subset M of $\mathbb{H}$ the set*
$M^{\perp} = \{\mathbf{v} \in \mathbb{H} \mid \quad (\mathbf{u}, \mathbf{v}) = 0 \quad \forall \mathbf{u} \in M\}$ *is called the* ***orthogonal space*** *with respect to M.*

Definition 70 *Let $\mathbb{V}$, $\mathbb{W} \subset \mathbb{H}$ be closed subspaces of the HILBERT space $\mathbb{H}$. If every $\mathbf{u} \in \mathbb{H}$ is uniquely described as the sum $\mathbf{u} = \mathbf{v} + \mathbf{w}$ with $\mathbf{v} \in \mathbb{V}$ and $\mathbf{w} \in \mathbb{W}$ then $\mathbb{H}$ is called the* ***direct sum*** *of the subspaces $\mathbb{V}$ and $\mathbb{W}$:*
$\mathbb{H} = \mathbb{V} \oplus \mathbb{W}$.

Definition 71 $\mathbb{W} \subset \mathbb{H}$ *is called an* ***orthogonal complement*** *of the closed subspace* $\mathbb{V} \subset \mathbb{H}$ *if and only if* $\mathbb{W} \perp \mathbb{V} \quad \wedge \quad \mathbb{H} = \mathbb{V} \oplus \mathbb{W}$.

Theorem 72 *Let* $\mathbb{W}$ *be the orthogonal complement of* $\mathbb{V}$*, that means* $\mathbb{W} \perp \mathbb{V} \quad \wedge \quad \mathbb{H} = \mathbb{V} \oplus \mathbb{W}$, $\mathbb{H}$*: HILBERT space. Then every* $\mathbf{u} \in \mathbb{H}$ *may be decomposed uniquely into the sum* $\mathbf{u} = \mathbf{v} + \mathbf{w}$ *of an element* $\mathbf{v} \in \mathbb{V}$ *and an element* $\mathbf{w} \in \mathbb{W}$ *such that* $(\mathbf{v}, \mathbf{w}) = 0$.

$\mathbf{v}$ *is called the* ***orthogonal projection*** *of* $\mathbf{u}$ *onto* $\mathbb{V}$ *and* $\mathbf{w}$ *is called the* ***orthogonal projection*** *of* $\mathbf{u}$ *onto* $\mathbb{W}$*.*
The mappings $\mathbf{P} : \mathbb{H} \to \mathbb{V}$ *with* $\mathbf{Pu} = \mathbf{v}$ *and* $\mathbf{Q} : \mathbb{H} \to \mathbb{W}$ *with* $\mathbf{Qu} = \mathbf{w}$ *are called* ***orthogonal projectors*** *(Orthoprojector) onto* $\mathbb{V}$ *or onto* $\mathbb{W}$*.*

Theorem 73 *Let* $\mathbb{U}$ *be a closed subspace of the HILBERT space* $\mathbb{H}$ *and* $\mathbf{u}$ *be an arbitrary element of* $\mathbb{H}$*.*
There exists a unique element $\mathbf{u}_0 \in \mathbb{U}$ *with*

$$\begin{aligned} a)\ \|\mathbf{u} - \mathbf{u}_0\| &= \min_{\mathbf{v} \in \mathbf{U}} \|\mathbf{u} - \mathbf{v}\| \ \textit{and} \\ b)\ (\mathbf{u} - \mathbf{u}_0, \mathbf{v}) &= 0 \qquad for \quad \forall \mathbf{v} \in \mathbb{U}, \quad i.e.\ \mathbf{u} - \mathbf{u}_0 \in \mathbb{U}^{\perp}. \end{aligned}$$

$\mathbf{u}_0$ *is called the* ***best approximation of*** $\mathbf{u} \in \mathbb{H}$ *with respect to the subspace* $\mathbb{U}$*.*
Proof: see literature (Kantorowitsch/Akilow)

3.1 Linear Operators in HILBERT Spaces

Definition 74 *Given a linear operator in the HILBERT space* $\mathbb{H}$ *with the domain* $D(\mathbf{A}) \subseteq \mathbb{H}$ *and the range* $\mathbb{H}$*;* $\overline{D(\mathbf{A})} = \mathbb{H}$*. The set*

$$D(\mathbf{A}^*) = \{\mathbf{x} \in \mathbb{H} \mid \quad \exists \mathbf{y} \in \mathbb{H} \ \textit{with}\ (\mathbf{Au}, \mathbf{x}) = (\mathbf{u}, \mathbf{y}) \quad \forall \mathbf{u} \in D(\mathbf{A})\}$$

is a subset of $\mathbb{H}$*. Then the operator* $\mathbf{A}^* : D(\mathbf{A}^*) \subseteq \mathbb{H} \to \mathbb{H}$ *with* $\mathbf{A}^*\mathbf{x} = \mathbf{y}$ *is called the* ***adjoint operator*** *(or Hermitian conjugate) of* $\mathbf{A}$*. Thus*

$$(\mathbf{Au}, \mathbf{x}) = (\mathbf{u}, \mathbf{A}^*\mathbf{x}) \ for\ \forall \mathbf{u} \in D(\mathbf{A}), \forall \mathbf{x} \in D(\mathbf{A}^*).$$

Definition 75 *Let* $\mathbb{H}$ *be a HILBERT space. Given the linear operator* $\mathbf{A} : D(\mathbf{A}) \subseteq \mathbb{H} \to \mathbb{H}$ *with* $\overline{D(\mathbf{A})} = \mathbb{H}$*.* $\mathbf{A}$ *is called*

- **symmetric** $\iff (\mathbf{Au}, \mathbf{x}) = (\mathbf{u}, \mathbf{Ax}) \quad \forall \mathbf{u}, \mathbf{x} \in D(\mathbf{A})$,
  i.e. $\mathbf{A}^*\mathbf{x} = \mathbf{Ax} \quad \forall \mathbf{x} \in D(\mathbf{A}) \quad \wedge \quad D(\mathbf{A}) \subseteq D(\mathbf{A}^*)$
- **selfadjoint** $\iff \mathbf{A} = \mathbf{A}^*$, i.e. $\mathbf{A}^*\mathbf{x} = \mathbf{Ax} \quad \forall \mathbf{x} \in D(\mathbf{A}) = D(\mathbf{A}^*)$

- **skew symmetric** $\iff$ $(\mathbf{Au},\mathbf{x}) = -(\mathbf{u},\mathbf{Ax}) \quad \forall \mathbf{u},\mathbf{x} \in D(\mathbf{A})$
- **skew adjoint** $\iff$ $\mathbf{A} = -\mathbf{A}^*$

Definition 76 *The linear operator* $\mathbf{A} : D(\mathbf{A}) \subseteq \mathbb{H} \to \mathbb{H}$ *is called* ***positive definite****, if and only if:*
$(\mathbf{Au},\mathbf{u}) \geq C \left\|\mathbf{u}\right\|^2$ *for* $\forall \mathbf{u} \in D(\mathbf{A}), \quad C \in \mathbb{R}, \quad C > 0$

Definition 77 *The complex number* $\lambda \in \mathbb{C}$ *is called the* ***eigenvalue of the operator*** $\mathbf{A} : \mathbb{H} \to \mathbb{H}$ *,if there exists an element* $\mathbf{x} \in \mathbb{H}$ *,* $\mathbf{x} \neq \mathbf{0}$*, such that* $\mathbf{Ax} = \lambda\mathbf{x}$*. Every* $\mathbf{x} \neq \mathbf{0}$ *with* $\mathbf{Ax} = \lambda\mathbf{x}$ *is called an* ***eigenelement*** *of the eigenvalue* λ*.*

Definition 78 *Operators in a HILBERT space* $\mathbb{H}$ *over the field* $\mathbb{K}$ *(*$\mathbb{R}$ *or* $\mathbb{C}$*) which map from* $\mathbb{H}$ *to* $\mathbb{K}$ *are called* ***functionals*** *or* ***linear forms****:*
$\mathbf{f} : \mathbb{H} \to \mathbb{K}$.

Definition 79 *The functional* $\mathbf{f} : \mathbb{H} \to \mathbb{K}$ *is called:*

- **linear***, if*: $\mathbf{f}(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha f(\mathbf{u}) + \beta f(\mathbf{v}) \quad \forall \mathbf{u},\mathbf{v} \in \mathbb{H};\ \forall \alpha,\beta \in \mathbb{K}$
- **bounded**, *if:* $\exists M \in \mathbb{R},\ M > 0 \quad | \quad |f(\mathbf{u})| \leq M \left\|\mathbf{u}\right\| \quad \forall \mathbf{u} \in \mathbb{H}$
- **continuous**, *if:* $\lim_{n\to\infty} \mathbf{u}_n = \mathbf{u}$ implies $\lim_{n\to\infty} \mathbf{f}(\mathbf{u}_n) = \mathbf{f}(\mathbf{u})$;
  $\mathbf{u}_n, \mathbf{u} \in \mathbb{H}$.

Definition 80 *Given a linear bounded functional* $\mathbf{f} : \mathbb{H} \to \mathbb{K}$*. The number*

$$\begin{aligned} \left\|\mathbf{f}\right\| &= \sup_{\|\mathbf{u}\|=1} |\mathbf{f}(\mathbf{u})|, \quad \mathbf{u} \in \mathbb{H} \\ &= \inf\{M \in \mathbb{R} \mid |\mathbf{f}(\mathbf{u})| \leq M \left\|\mathbf{u}\right\| \quad \forall \mathbf{u} \in \mathbb{H}\} \end{aligned}$$

is called the ***norm of the functional.***

Theorem 81 ***The HILBERT space representation theorem (RIESZ)***
Let $\mathbb{H}$ *be a HILBERT space* $\mathbb{H}$ *over the field* $\mathbb{K}$ *with the inner product* $(.,.)$ *and let* $\mathbf{f}(\mathbf{x})$ *be a continuous linear functional in* $\mathbb{H}$*. Then there exists a fixed unique element* $\mathbf{u}_0 \in \mathbb{H}$ *such that* $\mathbf{f}(\mathbf{x}) = (\mathbf{x},\mathbf{u}_0) \quad \forall \mathbf{x} \in \mathbb{H}$ *and* $\left\|\mathbf{f}\right\| = \left\|\mathbf{u}_0\right\|$.

Definition 82 *Given a sequence* $\{\mathbf{x}_n\}_{n=1}^{\infty}$ *in the HILBERT space* $\mathbb{H}$*.*
If $\lim_{n\to\infty} \mathbf{f}(\mathbf{x}_n) = \mathbf{f}(\mathbf{x})$ *for every linear continuous functional* $\mathbf{f}$ *on* $\mathbb{H}$ *then* $\{\mathbf{x}_n\}_{n=1}^{\infty}$ ***tends weakly*** *to* $\mathbf{x} \in \mathbb{H}$ *as* n *tends to infinity We write:* $\mathbf{x}_n \rightharpoonup \mathbf{x}$

Definition 83 *A **bilinear form** on a HILBERT space* $\mathbb{H}$ *over the field* $\mathbb{K}$ *is a bilinear mapping* $\mathbf{a}:\mathbb{H}\times\mathbb{H}\to\mathbb{K}$. *That means:*

$$\begin{aligned}\mathbf{a}(\alpha\mathbf{u}+\beta\mathbf{v},\mathbf{w}) &= \alpha\mathbf{a}(\mathbf{u},\mathbf{w})+\beta\mathbf{a}(\mathbf{v},\mathbf{w})\\ \mathbf{a}(\mathbf{w},\alpha\mathbf{u}+\beta\mathbf{v}) &= \alpha\mathbf{a}(\mathbf{w},\mathbf{u})+\beta\mathbf{a}(\mathbf{w},\mathbf{v})\quad\forall\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{H},\ \forall\alpha,\beta\in\mathbb{K}.\end{aligned}$$

Definition 84 *The bilinear form* $\mathbf{a}:\mathbb{H}\times\mathbb{H}\to\mathbb{K}$ *is called*

- **bounded**, if $\exists\ C>0;\ C\in\mathbb{R}\quad|\quad|\mathbf{a}(\mathbf{u},\mathbf{v})|\le C\,\|\mathbf{u}\|\,\|\mathbf{v}\|\quad\forall\mathbf{u},\mathbf{v}\in\mathbb{H}$
- **symmetric**, if $\mathbf{a}(\mathbf{u},\mathbf{v})=\mathbf{a}(\mathbf{v},\mathbf{u})\quad\forall\mathbf{u},\mathbf{v}\in\mathbb{H}$
- **positive semidefinite**, if $\mathbf{a}(\mathbf{u},\mathbf{u})\ge 0\quad\forall\mathbf{u}\in\mathbb{H}$
- **positive definite**, if $\mathbf{a}(\mathbf{u},\mathbf{u})\ge C\,\|\mathbf{u}\|^2\quad\forall\mathbf{u}\in\mathbb{H}\wedge C>0,\ C=const.$

3.2 The space $\mathbb{L}_2(a,b)$

Elements: measurable functions $\mathbf{f}:\ (a,b)\to\mathbb{C}$, with $(L)\int\limits_a^b|\mathbf{f}(t)|^2\,dt<\infty$

These functions are called square-integrable.
Arithmetic operations: $\mathbf{f}(t)+\mathbf{g}(t)$ and. $\lambda\mathbf{f}(t),\ \lambda\in\mathbb{C}$ are calculated pointwise

Inner product/Norm: $(\mathbf{f},\mathbf{g})=(L)\int\limits_a^b\mathbf{f}(t)\overline{\mathbf{g}(t)}dt,\quad\|\mathbf{f}\|^2=(L)\int\limits_a^b|\mathbf{f}(t)|^2\,dt$

SCHWARTZ's inequality: $|(\mathbf{f},\mathbf{g})|\le\|\mathbf{f}\|\,\|\mathbf{g}\|$, i.e.

$$\left|(L)\int\limits_a^b\mathbf{f}(t)\overline{\mathbf{g}(t)}dt\right|^2\le\left((L)\int\limits_a^b|\mathbf{f}(t)|^2\,dt\right)\cdot\left((L)\int\limits_a^b|\mathbf{g}(t)|^2\,dt\right)$$

Basis systems:

A) LEGENDRE polynomials

B) ONS with trigonometric functions

C) $(a,b)=(0,T)$: complete ONS:

$$\boldsymbol{\varphi}_k(t)=\frac{1}{\sqrt{T}}\exp(ik\omega t)\qquad k=0,\pm1,\pm2,...\qquad\omega=\frac{2\pi}{T}$$

Properties:

- The space $\mathbb{L}_2(a, b)$ is infinite-dimensional.

- $\mathbb{L}_2(a, b)$ is complete and separable.

- The elements of $\mathbb{L}_2(a, b)$ are classes of functions. $\mathbf{f}_1(t)$ and $\mathbf{f}_2(t)$ belong to the same class if $\mathbf{f}_1(t) = \mathbf{f}_2(t)$ almost everywhere over (a, b), i.e. if

  $\mathbf{f}_1(t) \neq \mathbf{f}_2(t)$ on a set of measure zero or $(L)\displaystyle\int_a^b |\mathbf{f}_1(t) - \mathbf{f}_2(t)|\, dt = 0$

Definition 85 $L_2(\mathbb{R}/2\pi) = \{f : \mathbb{R} \to \mathbb{C} \mid f(t)$ *is measurable,* $f(t + 2\pi) = f(t)\ \forall t \in R,$
$\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2 dt < \infty\Big\}$