Domination in graphs with application to network reliability

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Thesis

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(Dr.rer.nat.)

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Abstract

In this thesis we investigate different domination-related graph polynomials, like the connected domination polynomial, the independent domination polynomial, and the total domination polynomial. We prove some basic properties of these polynomials and obtain formulas for the calculation in special graph classes. Furthermore, we also prove results about the calculation of the different graph polynomials in product graphs and different representations of the graph polynomials.

One focus of this thesis lays on the generalization of domination-related polynomials. In this context the trivariate domination polynomial is defined and some results about the bipartition polynomial, which is also a generalization of the domination polynomial, is presented. These two polynomials have many useful properties and interesting connections to other graph polynomials. Furthermore, some more general domination-related polynomials are defined in this thesis, which shows some possible directions for further research.

Acknowledgements

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A special thanks goes to my family. Words cannot express how grateful I am to my mother and my brother for all of the support in the last years. Your prayers for me was what sustained me thus far. But especially I would like express appreciation to my beloved wife Ester who was always my support in the past years.

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1 Introduction

Technical systems like communication networks, power grids, traffic management systems, and enterprise data networks have a net-like structure. The mathematical model for such networks is an undirected or directed graph. Let us use a computer network as an example. Here the computers are the components (vertices) of the graph and the links between them represent the edges (e.g. see Figure 1.1). Now imagine, we want to monitor the functions of each of the computers by one or a small number of computers in such a way that every of these computers can control its neighbors. In graph theory we call these controllers a dominating (vertex) set. Several questions emerge naturally in this context. When the effort of monitoring the network is to be minimized, then a smallest dominating set provides the solution. What is the minimum number of computers required to completely monitor the communication network? How can we find such a minimum dominating set?

Now assume that the computers (routers, terminals) of the network are subject to random failure, but also failed computers have to be monitored. Then we can ask for the probability that the complete network is monitored. Translated to the language of graph theory, we ask for the probability that a randomly selected vertex subset forms a dominating set in the graph that reflects the topology of the given computer network.

A variety of new problems appears as soon as we impose additional properties on the dominating set. We can, for instance, restrict the choice of dominating sets to independent vertex sets or to vertex sets that induce connected subgraphs. The last requirement has interesting applications for routing in wireless networks.
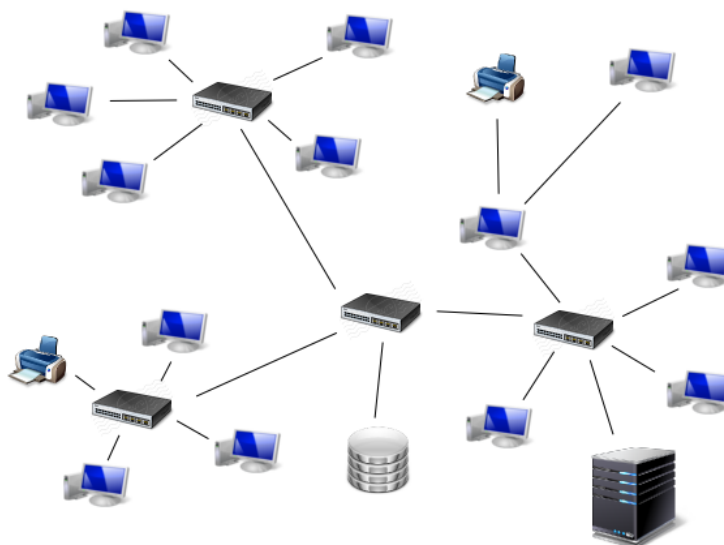
Fig. **1.1**: A computer network.

The calculation of discrete probabilities leads directly to problems of counting and enumeration. The computation of the probability of a given graph property in a finite random graph usually results in a subgraph counting problem. A classical example is the all-terminal reliability of networks which can be obtained from the number of connected spanning subgraphs. Generating functions for the counting sequences, i.e. graph polynomials, provide a powerful tool in the area of graphical enumeration. Unfortunately, the computation of many interesting graph polynomials, especially those ones considered in this thesis, is proved to be NP-hard.

The first graph polynomial studied in literature is the chromatic polynomial, which counts the number of proper colorings of graphs (a coloring of the vertices such that neighbors do not have the same color). It was defined by G.D. Birkhoff [Bir12] to attack the four color problem in 1912. This and other famous graph polynomials, like the Tutte polynomial, were extensively studied in the last hundred years. J.L. Arocha and B. Llano defined the domination polynomial in 2000 [AL00], which is the ordinary generating function for the number of dominating sets in a graph. Since then, a lot of papers were published about this polynomial and many results have been proved, like the calculation in special graph classes, or the location of the zeros of the domination polynomial.

The main aim of this dissertation is to investigate domination-related polynomials. This goal is divided into two parts. The first part is dedicated to the counting of dominating sets with special properties, e.g. they must be connected or independent. In the second part we introduce generalizations of several simple graph polynomials to more complex graph polynomials. These generalizations often give us new results and knowledge about the included graph polynomials. They also help us to find connections between the different graph polynomials and give some sort of organization of the various graph polynomials. This approach was mainly motivated by the paper of J. A. Makowsky with the title "From a Zoo to a Zoology: Towards a General Theory of Graph Polynomials" [Mak07].

As mentioned before, counting problems in graphs often have a connection to problems in the reliability context. The results about domination-related counting problems in this thesis can also be applied to the corresponding reliability polynomials.

1.1 Own Contributions and Publications

My own contributions are the definition of the connected domination polynomial, the trivariate domination polynomial and the independent domination polynomial. Furthermore, I proved several results about the different domination related polynomials, especially the calculation in product graphs, different recurrence equations and representations. Some of these results are already published or submitted:

- Markus Dod: *The Independent Domination Polynomial*, submitted to Graphs and Combinatorics.

- Markus Dod: *Graph products of the trivariate total domination polynomial and related polynomials*, submitted to Discrete Applied Mathematics.

- Markus Dod, Peter Tittmann: *Isolated Vertices in Random Graphs* in Scientific Reports, E-Mobility and Reliability in Communication Networks. Journal of the University of Applied Sciences Mittweida, 1 (2015).

- Markus Dod, Tomer Kotek, James Preen, and Peter Tittmann: *Bipartition polynomials, the ising model, and domination in graphs.* Discussiones Mathematicae Graph Theory 35.2 (2015), pp. 335–353.

- Markus Dod: *The total domination polynomial.* Congressus Numerantium 219 (2014), pp. 207–226.

The results were also partially presented in talks on international conferences:

- *On the counting of independent dominating sets*, Colloquium on Combinatorics, Ilmenau, Germany, 2014.

- *Graph products of the trivariate total domination polynomial*, 9th International colloquium on graph theory and combinatorics, Grenoble, France, 2014.

- *The Total Domination Polynomial*, Forty-Fifth Southeastern International Conference on Combinatorics, Graph Theory, and Computing, Boca Raton, USA, 2014.

- *The Total Domination Polynomial*, Colloquium on Combinatorics, Ilmenau, Germany, 2013.

- *The Bipartition Polynomial, the Ising Model and the Matching Polynomial*, CID 2013 - Colourings, Independence and Domination, Szklarska Poręba, Poland, 2013.

- *Generalisations of the domination reliability polynomial*, SEG-Workshop, Freiberg, Germany, 2012.

1.2 Organization of this Thesis

The thesis is organized as follows. In Chapter 2 some basic graph theoretic definitions and notations are introduced. In the following chapter the domination and the domination reliability polynomial are defined, a short overview over known results is given and some new results, especially for product graphs, are proved. Furthermore, the connection to the neighborhood polynomial is shown. The independent domination polynomial is defined in Chapter 4. We show some properties of the independent domination polynomial and prove formulas for the calculation in special graph classes and in product graphs.

In Chapter 5 some properties of the total domination polynomial are proved. Additionally, in this chapter the trivariate domination polynomial is defined. The main results are its connection to other graph polynomials and the calculation in product graphs. The connected domination polynomial is defined in Chapter 6. Chapter 7 presents some results about the bipartition polynomial, especially about the encoded graph invariants and the calculation in special graph classes. Furthermore, we present some results for the counting of bipartite subgraphs in this chapter, which yields a connection to the edge-cover polynomial. Chapter 8 shows some possible further generalizations of the domination polynomial and Chapter 9 summarizes this thesis.

2 Basics

In this chapter we introduce the basic graph theoretic concepts and definitions that we use in this thesis. The definitions follow the standard definitions of the textbooks [Har69; Die00; Wes01] and [GY04].

Definition 2.1. *A graph $G = (V, E)$ is an ordered pair of a finite set of vertices V and a set of edges E, such that every edge is an one- or two-element subset of the vertex set. We call an edge $e \in E$ a* loop *if it is an one-element subset of V. A graph is called* simple *if it has no loops and it is called* nontrivial *if $|V| \geq 2$.*

In the following we write $V(G)$ and $E(G)$ for the vertex and the edge set of the graph G and if there is no risk of confusion we simply write V and E. The *order* of a graph denotes the number of vertices and the *size* its number of edges. A vertex $u \in V$ is called *adjacent* to $v \in V$ if $\{u, v\} \in E$ and two edges are called adjacent if they have one end vertex in common. An edge $\{u, v\} \in E$ is called *incident* to its end vertices u and v. Let v be a vertex of $G = (V, E)$, then $N(v) = \{u \in V : \{u, v\} \in E\}$ is the *open neighborhood* of the vertex v and $N[v] = N(v) \cup \{v\}$ is the *closed neighborhood*. Let $U \subseteq V$ be a vertex subset, then

$$N(U) = \bigcup_{u \in U} N(u) \backslash U$$

is the *open neighborhood* of the subset U. Sometimes we need the union of the open neighborhoods of the vertices in U. This set is called the *total open neighborhood*

$$N^t(U) = \bigcup_{v \in U} N(v).$$

Let now $U \subseteq V$ and $u \in U$. Then $\mathrm{PN}(u, U)$ denotes the set of *private neighbors* of u with respect to the vertex subset U. A vertex $v \in V$ is called a *private neighbor* of u if $N[v] \cap U = \{u\}$. Note that if $u \in U$ is not adjacent to any other vertex in U, then $u \in \mathrm{PN}(u, U)$.

According to the definition of the open neighborhood of a vertex, the *degree* of a vertex is denoted by $\deg_G(v)$ and it is the size of the open neighborhood of this vertex. The minimum and maximum degree of a graph are given by $\delta(G) = \min_{v \in V}\{\deg(v)\}$ and $\Delta(G) = \max_{v \in V}\{\deg(v)\}$. Furthermore, we denote by $\sharp \deg_i(G)$ the number of vertices with degree i of the graph G.

A *path* in a graph $G = (V, E)$ is an ordered sequence of pairwise different vertices $v_1, \ldots, v_n$, such that $\{v_i, v_{i+1}\} \in E$, for $i \in \{1, \ldots, n-1\}$. A graph G is *connected* if at least one path between every pair of its vertices exists.

The *complement* $\bar{G}$ of a graph $G = (V, E)$ has the vertex set V and two vertices in $\bar{G}$ are adjacent if and only if these two vertices are non-adjacent in G. The *line-graph* $L(G)$ of a graph G has a vertex for each edge of G and two vertices of $L(G)$ are adjacent if and only if they correspond to two adjacent edges in G.

Definition 2.2. *Let $G = (V, E)$ and $G' = (V', E')$ be graphs. Then G' is called a* subgraph *of G ($G' \subseteq G$), if $V' \subseteq V$ and $E' \subseteq E$.*

Definition 2.3. *Let $G = (V, E)$ be a graph and $F \subseteq E$ be an edge subset. Then the* spanning *subgraph $G\langle F \rangle$ is the graph*

$$G\langle F \rangle = (V, F).$$

Definition 2.4. *Let $G = (V, E)$ be a graph and $U \subseteq V$ be a vertex subset of G. The* induced *subgraph $G[U]$ of G is the graph*

$$G[U] = (U, \{e \in E | e \subseteq U\}).$$

In short, we write $E(U)$ for the edge subset $\{e \in E | e \subseteq U\}$.

Definition 2.5. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be simple graphs. An* isomorphism *from G to H is a bijection $\phi : V(G) \to V(H)$ such that $\{u, v\} \in E(G)$ if and only if $\{\phi(u), \phi(v)\} \in E(H)$. If such an isomorphism for two graphs G and H exists, we write $G \cong H$.*

Let $\mathcal{H}$ be a set of graphs and G be a graph. The graph G is called $\mathcal{H}$-*free* if no induced subgraph of G is isomorphic to a graph in $\mathcal{H}$. If $\mathcal{H}$ is a specific graph class (e.g. cycles or paths), then we simply write that the graph G does not *contain* this graph class. Similar to the definition of the (vertex-) induced subgraph of G we can define the edge-induced subgraph for a given edge subset.

Definition 2.6. *Let $G = (V, E)$ be a graph and $F \subseteq E$ be an edge subset of G. The* edge-induced *subgraph $G[F]$ of G is the graph*

$$G[F] = \left(\bigcup_{e \in F} e, F \right).$$

A maximum connected subgraph of G is called a *component* of G and a component is called *covered* if it contains at least one edge. A vertex of degree zero is called an *isolated* vertex.

Definition 2.7. *Let $G = (V, E)$ be a graph, then $k(G)$ denotes the number of components, $c(G)$ the number of covered components and $\mathrm{iso}(G)$ denotes the number of isolated vertices of G. Therefore, $k(G) = c(G) + \mathrm{iso}(G)$. Furthermore, $\mathrm{Comp}(G)$ denotes the set of covered components of G.*

Definition 2.8. *[KPT13] Let $G = (V, E)$ be a graph with k components of size λ_i, $i \in \{1, \ldots, k\}$, and $|V| = n$. The* type *of G is the integer partition $\lambda_G = (\lambda_1, \ldots, \lambda_k) \vdash n$. We write $i \in \lambda_G$ in order to indicate that i is a part of λ_G. The number of parts is denoted by $|\lambda_G|$.*

Definition 2.9. *Let $G = (V, E)$ be a graph. A* hole *in the graph G is a cordless cycle of length at least four and an* anti-hole *is the complement of such a cycle.*

A vertex subset is called *independent* if its vertices are pairwise non-adjacent and it is called a *clique* if its vertices are pairwise adjacent. We denote by $\alpha(G)$ and $\omega(G)$ the order of a maximum independent subgraph and the order of a maximum clique, respectively. A *matching* of the graph G is an edge subset F such that $e \cap f = \emptyset$, for all $e, f \in F$ with $e \neq f$. A matching F is called *perfect* if $\bigcup_{e \in F} e = V$. A (vertex) *coloring* of the graph $G = (V, E)$ with k colors is a function $c : V \to \{1, \ldots, k\}$ and the coloring c is called *proper* if $c(u) \neq c(v)$, for all $\{u, v\} \in E$.

Definition 2.10. *Let $G = (V, E)$ be a graph and $X \subseteq V$ be a vertex subset. Then X is called* a separating vertex set *if the graph $G - X$ has at least two components. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two subgraphs of G with $V_1 \cap V_2 = X$, $V_1 \cup V_2 = V$, $E_1 \cap E_2 = \emptyset$ and $E_1 \cup E_2 = E$, then we call (G_1, G_2, X) a* splitting *of G. If $|X| = 1$, then the separating vertex is called an* articulation.

2.1 Graph Operations

In this thesis we use a diversity of graph operations, that can partially be found in the literature. Let $v \in V$ be a vertex and $e = \{u, v\} \in E$ be an edge of G.

- $G - v$ denotes the graph obtained from G by the removal of v and all edges incident to v.

- G/v denotes the graph obtained from G by the removal of v and the addition of edges between any pair of non-adjacent vertices of $N(v)$.

- $G \odot v$ denotes the graph obtained from G by removing all edges between vertices of $N(v)$.

- $G \circledcirc v$ denotes the graph $G \odot v - v$.

- $G \circ v$ denotes the graph obtained from G by removing v and adding loops to all neighbors of v.

- $G - X$ denotes the graph obtained by deleting all vertices of the vertex subset $X \subseteq V$ and the edges incident to them.

- $G \backslash X$ denotes the fusion of all vertices of X to a single vertex. Furthermore, $G \backslash_x X$ denotes the fusion of all vertices of X and the new vertex is labeled with x.

- $G \rhd\!\!-\!\!\circ X$ denotes the fusion of all vertices of $X \subseteq V$ and the addition of a new vertex which is adjacent to the fused one.

- $G + \{v, \cdot\}$ denotes the graph $(V \cup \{v'\}, E \cup \{v, v'\})$ obtained from G by adding a new vertex v' and an edge $\{v, v'\}$ to G.

- $G + \{X, \cdot\}_u$ denotes the graph $(V \cup \{u\}, E \cup \{\{u, x\} : x \in X\})$ obtained from G by adding a new vertex u and edges joining all vertices of X with u.

- $G - e$ denotes the graph obtained from G by removing e.

- G/e (contraction of the edge e) denotes the graph obtained from G by removing e and unifying the end vertices of e.

- $G\dagger e$ (extraction of the edge $e = \{u, v\}$) denotes the graph $G - u - v$.

Composite operations will be applied from left to right, e.g. $G - e/u$ which stands for $(G - e)/u$.

Remark 2.11. *Let $G = (V, E)$ be a graph and $v \in V$. Then*

$$(G \odot v)/v \cong G/v,$$
$$(G \odot v) - N[v] \cong G - N[v]. \tag{2.1}$$

2.2 Graph Classes

In this section, several graph classes will be defined which are used in this thesis. For more graph classes see the "Information System on Graph Classes and their Inclusions" [Rid+01].

A graph G is called *complete* if all vertices are pairwise adjacent. We denote a complete graph with n vertices by K_n and the edgeless graph with n vertices by E_n. Note that $V(K_n) = \{1, \ldots, n\}$. If we remove only a few edges from the complete graph we obtain the so called k-bounded complete graphs.

Definition 2.12. *A graph $G = (V, E)$ is k-bounded complete if every vertex in V has at most k non-neighbors in V.*

Remark 2.13. *The 1-bounded complete graphs are obtained from a complete graph by removing a matching.*

Definition 2.14. *Let k and l be two natural numbers, K_{n_0} be a complete graph with n_0 vertices and E_{n_i} be an edgeless graph of order n_i, for $2 \le n_i \le k + 1$, $i \in \{1, \ldots, l\}$, and $n = n_0 + n_1 + \cdots + n_l$. Then the simple k-bounded complete graph K_n^k of the type $\Lambda(K_n^k) = [n_0, n_1, n_2, \ldots, n_l]$ is the join (see Definition 2.35) of these $l + 1$ graphs.*
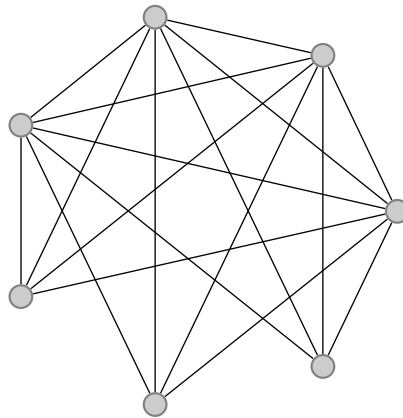
Fig. 2.1: Simple 2-bounded complete graph with 7 vertices.

A *bipartite graph* consists of two independent vertex sets X and Y and edges joining the vertices of these two sets. A bipartite graph is called *complete* (denoted by $K_{m,n}$ if $|X| = m$ and $|Y| = n$) if all vertices of X are adjacent to all vertices of Y. König [Kön36] proved the famous theorem which says that a graph is bipartite if and only if it has no odd cycle. Some special bipartite graphs are the k-bounded bipartite graphs.

Definition 2.15. *[Rid+01] A bipartite graph $G = (X \cup Y, E)$ is k-bounded bipartite $K_{m,n}^k$ if every vertex in X, respectively Y, has at most k non-neighbors in Y, respectively X.*

Remark 2.16. *The 1-bounded bipartite graphs are constructed from a complete bipartite graph by removing a matching.*

Let $G = (V, E)$ be a graph and $v \in V$. If $N(v)$ induces a clique in G, then v is called *simplicial*. A *perfect elimination ordering* is an ordering $s = (v_1, \ldots, v_n)$ of V with the property that v_i is a simplicial vertex of $G[\{v_i, \ldots, v_n\}]$, for all $i \in \{1, \ldots, n\}$. A graph is called *chordal* if it has a perfect elimination ordering.
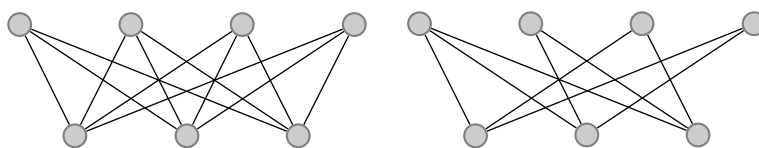
Fig. 2.2: A complete bipartite graph $K_{4,3}$ and a 1-bounded bipartite graph $K_{4,3}^1$.

A *tree* is a connected graph without (induced) cycles and a *forest* is a graph such that all its components are trees. A generalization of trees are k-trees.

Definition 2.17. *The complete graph with k vertices is a k-tree. A k-tree with $n+1$ vertices $(n \geq k)$ can be constructed from a k-tree with n vertices by adding a vertex adjacent to all vertices of a k-clique of the existing k-tree.*
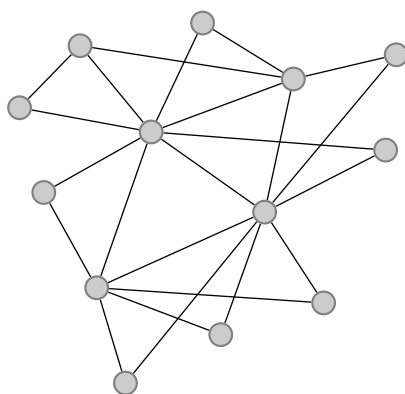
Fig. 2.3: A 2-tree with 13 vertices.

It follows directly from the definition of the k-tree that every k-tree is a chordal graph and therefore there exists a perfect elimination ordering. The class of k-trees can be characterized by a generalization of line graphs.

Definition 2.18. *[MJP06] The k-line graph of a graph G is defined as a graph whose vertices are the cliques of size k in G. Two vertices are adjacent in the k-line graph if and only if the corresponding cliques in G have $k-1$ vertices in common.*

Definition 2.19. *A graph $G = (V, E)$ is a* split graph *if the vertex set can be partitioned in an independent set and a clique. A split graph of order n is called an (n, k)-star $S_{n,k}$ if the clique has the order k and every vertex of the clique is adjacent to all vertices of the independent set.*

Lemma 2.20. *A k-tree $G = (V, E)$ is an (n, k)-star, with $n > k$, if and only if the $(k+1)$-line graph of G is a clique of size $n - k$.*

Proof. Each leaf vertex is a clique of size $k + 1$ together with the k center vertices and all of the $(k+1)$-cliques have k vertices in common. Therefore, all vertices in the $(k+1)$-line graph are pairwise adjacent.

If the $(k+1)$-line graph is a clique of size $n-k$, then the corresponding cliques have pairwise k vertices in common. Hence, the original graph is an (n, k)-star. □
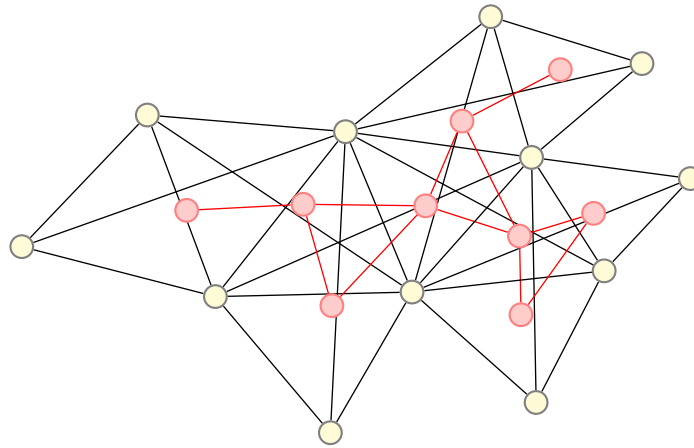
Fig. 2.4: A 3-tree and its 4-line graph.
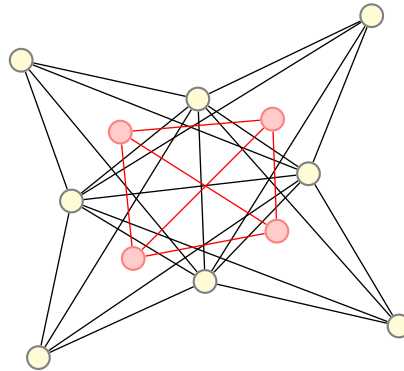
Fig. 2.5: A $(8,4)$-star and its 5-line graph.

Definition 2.21. *[MJP06] The complete graph with k vertices is a* simple k-tree. *A simple k-tree with $n+1$ vertices $(n \geq k+1)$ can be constructed from a simple k-tree with n vertices by adding a vertex adjacent to all vertices of a k-clique not previously chosen in the existing simple k-tree.*

Lemma 2.22. *[MJP06] A k-tree $G = (V,E)$ with $n > k$ vertices is a simple k-tree if and only if the $(k+1)$-line graph of G is a tree.*

Lemma 2.23. *[MJP06] Let G be a k-tree with $n > k$ vertices. G is a k-path graph if and only if G is a simple k-tree with exactly two simplicial vertices.*

Definition 2.24. *Let G be a k-path of order n with the vertex set $V = \{1,\dots,n\}$. Then the k-path is called* simple *or short $P_n^{(k)}$ if there exists a perfect elimination ordering $s = (1,2,\dots,n)$ with $N_{G[\{l,\dots,n\}]}(l) = \{l+1,\dots,l+k\}$, for all $l \in \{1,\dots,n-k\}$ (see Figure 2.6).*

Definition 2.25. *A k-cycle $C_n^{(k)}$ occurs from a simple k-path $P_n^{(k)}$ by adding edges between the first k vertices and the last k vertices of the simple k-path.*

Remark 2.26. *A path P_n is the simple 1-path $P_n^{(1)}$ and the cycle C_n is the 1-cycle $C_n^{(1)}$.*
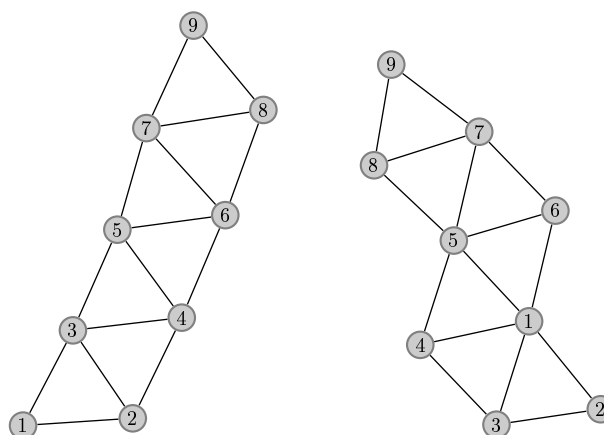
Fig. 2.6: A simple 2-path and a 2-path.
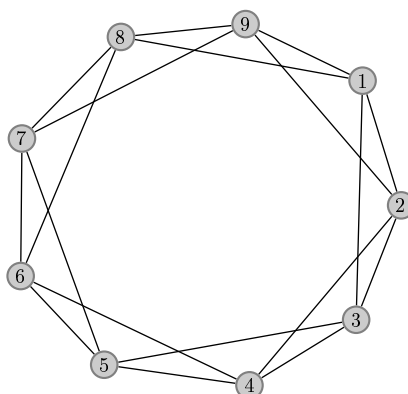
Fig. 2.7: A 2-cycle with 9 vertices.

2.3 Graph Products

Graph products are well known in literature and have many applications. Some enumeration and decision problems, e.g. finding the maximum number of non-attacking kings that can be paced on an $n \times m$-chessboard, have a natural relation to graph products. For a more detailed explanation and applications of graph products, see the "Handbook of product graphs" [HIK11]. For simplicity we assume that all graphs are nontrivial and simple. The vertex set of the following four products is the Cartesian product of the vertex sets of the two graphs, denoted by $V(G) \times V(H)$. However, each product has different rules to generate the edge set. The first product of interest is the Cartesian product of the two graphs G and H.

Definition 2.27. *The* Cartesian product $G \square H$ *of the graphs* $G = (V(G), E(G))$ *and* $H = (V(H), E(H))$ *is a graph such that*

1. *the vertex set of $G \square H$ is $V(G) \times V(H)$ and*

2. *two vertices (u, v) and (x, y) are adjacent in $G \square H$ if and only if $u = x$ and $\{v, y\} \in E(H)$ or $\{u, x\} \in E(G)$ and $v = y$.*

Theorem 2.28. *[HIK11] The graph $G \square H$ is connected if and only if G and H are connected.*

P_3
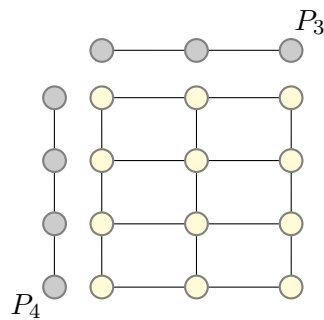
Fig. **2.8**: The Cartesian product $P_4 \square P_3$.

The second product of interest is the tensor product. In the literature this product has a lot of different names: Direct product, Kronecker product, cardinal product, relational product, cross product, conjunction, weak direct product, Cartesian product, product, and categorical product. See [HIK11; Wei62] for a detailed overview over the tensor product. Figure 2.9 shows the tensor product $P_4 \times P_3$.

Definition 2.29. *The* tensor product *or the* categorical product $G \times H$ *of the graphs* $G = (V(G), E(G))$ *and* $H = (V(H), E(H))$ *is a graph such that*

1. *the vertex set of* $G \times H$ *is* $V(G) \times V(H)$ *and*

2. *two vertices* (u, v) *and* (x, y) *are adjacent in* $G \times H$ *if and only if* $\{u, x\} \in E(G)$ *and* $\{v, y\} \in E(H)$.

P_3

P_4
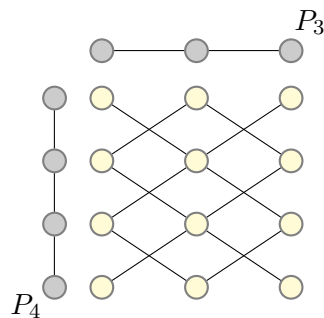
Fig. **2.9**: The tensor product $P_4 \times P_3$.

Theorem 2.30. *[Wei62] Let* $G = (V(G), E(G))$ *and* $H = (V(H), E(H))$ *be two graphs with* $|V(G)|, |V(H)| \geq 2$. *The graph* $G \times H$ *is connected if and only if* G *or* H *contains an odd cycle. If both graphs are connected and bipartite, then* $G \times H$ *has exactly two components.*

The lexicographic product was first introduced by Harary [Har59] as the composition of graphs and it is also known as graph substitution.

Definition 2.31. *The* lexicographic product $G \cdot H$ *of the graphs* $G = (V(G), E(G))$ *and* $H = (V(H), E(H))$ *is a graph such that*

1. *the vertex set of* $G \cdot H$ *is* $V(G) \times V(H)$ *and*

2. *two vertices (u, v) and (x, y) are adjacent in $G \cdot H$ if and only if $\{u, x\} \in E(G)$ or $u = x$ and $\{v, y\} \in E(H)$.*
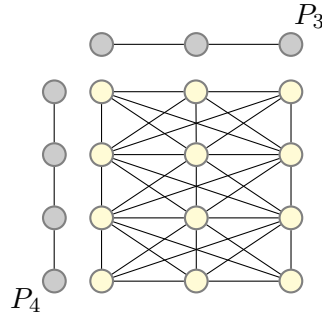
Fig. 2.10: The lexicographic product $P_4 \cdot P_3$.

Theorem 2.32. *[HIK11] The graph $G \cdot H$ is connected if and only if G is connected.*

Definition 2.33. *The* strong product *(or AND product) $G \boxtimes H$ of the graphs G and H is a graph such that*

1. *the vertex set of $G \boxtimes H$ is $V(G) \times V(H)$ and*

2. *$E(G \boxtimes H) = E(G \square H) \cup E(G \times H)$.*
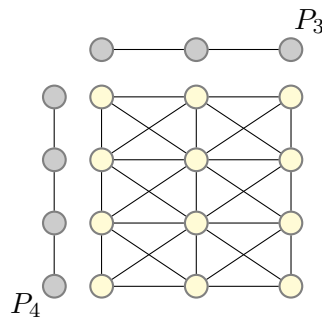
Fig. 2.11: The strong product $P_4 \boxtimes P_3$.

It follows immediately from the connection to the Cartesian product that the graph $G \boxtimes H$ is connected if and only if G and H are connected.

Theorem 2.34. *[HIK11] The Cartesian product, the tensor product and the strong product are commutative, associative and distributive. The lexicographic product is not commutative, but associative and right-distributive. Furthermore, the K_1 is an unit with respect to these four graph products.*

For proofs and explanations it is necessary to identify special vertices in the product graph. Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. As the Figures 2.8 - 2.11 show, we can draw the products of these two graphs in a grid structure. Therefore, the *row R_v* is the

vertex subset $\{(v, w) : w \in V(H)\}$ and the *column* C_w is the vertex subset $\{(v, w) : v \in V(G)\}$. If we talk about the vertices in the same row or column, then we mean the vertices in such a set. Let now G and H be two graphs with a linearly ordered vertex set. Then the first row of the product graph means the vertex subset $\{(\min(V(G)), w) : w \in V(H)\}$.

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two vertex disjoint graphs, then the *union* $G \cup H$ is the graph $(V(G) \cup V(H), E(G) \cup E(H))$.

Definition 2.35. *[Har69] The* join $G * H$ *of two graphs* $G = (V(G), E(G))$ *and* $H = (V(H), E(H))$ *is the graph union* $G \cup H$ *together with all the edges joining* $V(G)$ *and* $V(H)$.

With the join of two graph we are able to define the fan and the wheel.

Definition 2.36. *The join of a path* P_{n-1} *and the* K_1 *is called* fan F_n *(*$F_n \cong P_{n-1} * K_1$*) and the join of a cycle* C_{n-1} *with the* K_1 *is called* wheel W_n *(*$W_n \cong C_{n-1} * K_1$*). Furthermore, the join of edgeless graph* E_{n-1} *with the* K_1 *is called* star S_n *(*$S_n \cong E_{n-1} * K_1$*).*

Frucht and Harary introduced the corona of two graphs in 1970.

Definition 2.37. *[FH70] Let* $G = (V(G), E(G))$ *and* $H = (V(H), E(H))$ *be graphs. Then the* corona *of* G *and* H *is the graph* $G \circ H$ *which is the disjoint union of* G *and* $|V(G)|$ *copies of* H *and every vertex* v *of* G *is adjacent to every vertex in the corresponding copy of* H.
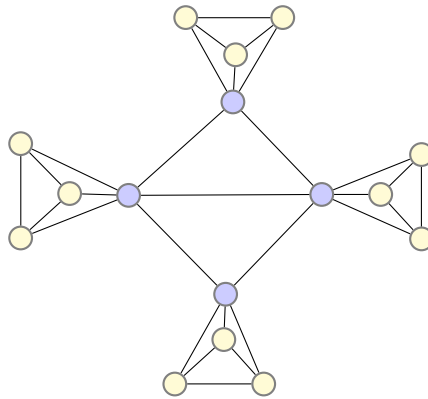
Fig. 2.12: Corona graph of the diamond and the K_3.

2.4 Graph Polynomials

Let $\mathcal{G}$ be the set of finite graphs and S some arbitrary set. Then *graph invariants* are functions $f : \mathcal{G} \to S$ such that for two graphs G and H

$$G \cong H \Rightarrow f(G) = f(H).$$

In case of S being equal to $\{0, 1\}$, we speak of *graph properties*, e.g. connectivity, and in the case of $S = \mathbb{N}$ of *graph parameters*, e.g. number of vertices or minimum degree. With the definition of the graph invariants we are able to define the graph polynomials.

Definition 2.38. *Let $\mathcal{G}$ be the set of finite graphs and $\mathbb{R}[x_1, \ldots x_k]$ the polynomial ring over the real numbers. Then a* graph polynomial *is a function $P : \mathcal{G} \to \mathbb{R}[x_1, \ldots x_k]$ such that for two graphs G and H*

$$G \cong H \Rightarrow P(G, x_1, \ldots, x_k) = P(H, x_1, \ldots, x_k).$$

In the following the coefficients of the graph polynomials are always integers. We denote by $\deg_{x_i}(P)$ the *degree* of the variable x_i of the graph polynomial $P(G, x_1, \ldots, x_k)$. If the polynomial has only one variable x, we write $\deg(P)$ instead of $\deg_x(P)$.

Definition 2.39. *Let $P = P(G, x_1, \ldots, x_k) = \sum\limits_{i_1, \ldots, i_k} a_{i_1, \ldots, i_k} x_1^{i_1} \ldots x_k^{i_k}$ be a graph polynomial, then*

$$[x_j^l]P = \sum_{\substack{i_1, \ldots, i_k \\ i_j = l}} a_{i_1, \ldots, i_k} x_1^{i_1} \ldots x_{j-1}^{i_{j-1}} x_{j+1}^{i_{j+1}} \ldots x_k^{i_k}.$$

If $P(G, x)$ is a graph polynomial with one variable x, then $[x^k]P(G, x)$ is the coefficient of x^k in the polynomial. Furthermore, we simply write $[x_{j_1}^{l_1} \ldots x_{j_l}^{l_l}]P(G, x_1, \ldots, x_k)$ instead of $[x_{j_1}^{l_1}](\ldots([x_{j_i}^{l_i}]P(G, x_1, \ldots, x_k))\ldots)$.

In the following a short overview over some graph polynomials used in this thesis is presented. A more detailed overview can be found in [Kot12; Tri12a]. Figure 9.1 shows the connection between different graph polynomials. But this figure shows only a part of the "graph of graph polynomials" (this phrase and some parts of the figure are introduced by M. Trinks [Tri12a]). The first graph polynomial studied in the literature is the chromatic polynomial $\chi(G, x)$. It was first defined by G. D. Birkhoff [Bir12] and yields the number of proper colorings of the graph with x colors.

Definition 2.40. *[DKT05] Let $G = (V, E)$ be a graph and $b_k(G)$ be the number of partitions of V in k independent vertex subsets. Then the* chromatic polynomial $\chi(G, x)$ *is defined as*

$$\chi(G, x) = \sum_{k=0}^{n} b_k(G) x^{\underline{k}},$$

where $x^{\underline{k}}$ is the falling factorial $x^{\underline{k}} = x(x-1) \ldots (x - (k-1))$.

The domination polynomial is the ordinary generating function for the number of dominating sets of the graph. Let $G = (V, E)$ be a graph and $W \subseteq V$ be a vertex subset of the graph. Then W is called a *dominating set* if and only if $N[W] = V$. The domination polynomial was introduced by J. Arocha and B. Liano [AL00].

Definition 2.41. *[AL00] Let $G = (V, E)$ be a graph. Then the* domination polynomial *is given by*

$$\mathrm{D}(G, x) = \sum_{\substack{W \subseteq V \\ N_G[W] = V}} x^{|W|}.$$

A vertex subset $X \subseteq V$ is a *vertex-cover* if $e \cap X \neq \emptyset$, for all $e \in E$.

Definition 2.42. *[Don+02] Let $G = (V, E)$ be a graph. Then the* vertex-cover polynomial $\Psi(G, x)$ *of G is defined as*

$$\Psi(G, x) = \sum_{\substack{X \subseteq V \\ X \text{ is vertex-cover}}} x^{|X|}.$$

Definition 2.43. *[LM05] Let $G = (V, E)$ be a graph. Then the* independence polynomial *$I(G, x)$ is the ordinary generating function for the number of independent sets of the graph:*

$$I(G, x) = \sum_{\substack{W \subseteq V \\ W \text{ is independent set}}} x^{|W|}.$$

Lemma 2.44. *[AO13] Let $G = (V, E)$ be a graph, $I(G, x)$ its independence polynomial and $\Psi(G, x)$ its vertex-cover polynomial. Then*

$$\Psi(G, x) = x^n I(G, 1/x).$$

An edge subset $F \subseteq E$ is an *edge-cover* if $\bigcup_{e \in F} e = V$.

Definition 2.45. *[AO13] Let $G = (V, E)$ be a graph. Then the* edge-cover polynomial *$E(G, z)$ of the graph is defined as*

$$E(G, z) = \sum_{\substack{F \subseteq E \\ F \text{ is edge-cover}}} z^{|F|}.$$

Definition 2.46. *[TAM11] Let $G = (V, E)$ be a graph. Then the* subgraph component *polynomial $\mathrm{Q}(G; v, x)$ is defined as*

$$\mathrm{Q}(G; v, x) = \sum_{W \subseteq V} v^{|W|} x^{k(G[W])}.$$

Let $G = (V, E)$ be a graph and $F \subseteq E$ be an edge subset, then *rank* of F is defined as $r(F) = |V| - c(G\langle F \rangle)$.

Definition 2.47. *[Tut67] Let $G = (V, E)$ be a graph. Then the* rank polynomial *$\mathrm{R}(G; x, y)$ is defined as*

$$\mathrm{R}(G; x, y) = \sum_{F \subseteq E} x^{r(F)} y^{|F| - r(F)}.$$

An other well-known polynomial is the Tutte polynomial.

Definition 2.48. *[Tut54; Tut67; Wel99] Let $G = (V, E)$ be a graph with n vertices. Then the* Tutte polynomial *is defined as*

$$\mathrm{T}(G; x, y) = \sum_{F \subseteq E} (x - 1)^{r(E) - r(F)} (y - 1)^{|F| - r(F)}.$$

The Tutte polynomial has a lot of connections to other graph polynomials (e.g. see [Wel99]). The next theorem shows two of these connections.

Theorem 2.49. *[Tut54; Tut67; Wel99] Let $G = (V, E)$ be a graph with n vertices. Then*

$$\mathrm{R}(G; x, y) = x^{n - c(G)} \mathrm{T}(G; x^{-1} + 1, y + 1)$$
$$\chi(G, x) = (-1)^{n - c(G)} x^{c(G)} \mathrm{T}(G; 1 - x, 0).$$

Definition 2.50. *[BT12] Let $G = (V, E)$ be a graph whose vertices fail independently of each other with a constant probability $1 - p$. Then the* residual network reliability *$R_1(G, p)$ is the probability that the surviving vertices induce a connected subgraph. The k-residual network reliability $R_k(G, p)$ is the probability that additionally at least k vertices are intact.*

2.5 Some Arrangements

We assume unless noted otherwise that all graphs in this thesis are simple. Let $G = (V, E)$ be a graph and $W \subseteq V$ be a dominating vertex subset. In such a case we call all vertices of W *dominating* and all vertices in the neighborhood of W *dominated*. Furthermore, sometimes we make a proof by case distinction with respect to a certain vertex v. In such a proof we simply say that the vertex v is either dominating or non-dominating, which means that we distinguish between dominating sets in which the vertex v is either contained or not.

Let $G = (V, E)$ be a graph and $X \subseteq V$ be a vertex subset of G. Then, occasionally, we can define a polynomial $f(G, x)$ under the condition that the vertices in X are already dominated. In this sense, *already dominated* means that we count in the polynomial $f(G, x)$ those vertex subsets of the graph $G + \{X, \cdot\}_u$ where the new vertex u is dominating and hence it dominates all vertices in X. In other words, let $A, B \subseteq V$ be vertex subsets, whereat only the vertices in A can be dominating and only the vertices in B must be dominated. In case of the domination polynomial we obtain:
$$\mathrm{D}(G, A, B; x) = \sum_{\substack{W \subseteq A \\ B \subseteq \bar{N}[W]}} x^{|W|}.$$

From this it follows that $\mathrm{D}(G, x) = \mathrm{D}(G, V, V; x)$. Furthermore, the domination polynomial under the condition that the vertex subset $X \subseteq V$ is already dominated is the polynomial $\mathrm{D}(G, V, V \setminus X; x)$. Analogously, this can be applied to other domination-related polynomials.

Furthermore, let 0^0 be equal one.

one step upwards in the tree and adding the father w

3 The Domination Polynomial

The domination polynomial was introduced by Arocha and Liano [AL00] (see Definition 2.41) and it is extensively studied in the literature. Let $d_i(G)$ be the number of the dominating sets of size i in G, then the domination polynomial can also be represented as

$$\mathrm{D}(G, x) = \sum_{i=1}^{n} d_i(G)x^i.$$

In general, the calculation of the domination polynomial is $\sharp W[2]$-complete [FG04] and therefore it is interesting to find graph classes in which the calculation can be done in polynomial time. In the Sections 3.1 and 3.2 such special graph classes, e.g. product graphs, are investigated. From the domination polynomial we can determine the size of a minimum dominating set in the graph G. The size of such a minimum set of the graph is called the *domination number* $\gamma(G)$ of the graph G, or shortly

$$\gamma(G) = min\{i : d_i(G) > 0\}.$$

The domination number was intensively studied in the literature. Several papers for the calculation of the domination number in general graphs and in special graph classes were published (e.g. see [GKL06; Gra06; VRB08; Roo11]). The first theorem shows the calculation of the domination polynomial of the join of two graphs.

Theorem 3.1. *[DT12] Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two vertex-disjoint graphs. Then the domination polynomial of the join of G and H can be calculated with*

$$\mathrm{D}(G * H, x) = \left((1+x)^{|V(G)|} - 1\right)\left((1+x)^{|V(H)|} - 1\right) + \mathrm{D}(G, x) + \mathrm{D}(H, x).$$

Another polynomial which is related to the domination polynomial is the neighborhood polynomial introduced by Brown and Nowakowski [BN08].

Definition 3.2. *[BN08] Let $G = (V, E)$ be a graph. Then the* neighborhood polynomial *is defined as*

$$\mathrm{N}(G, x) = \sum_{\substack{U \subseteq V \\ \exists u \in V \setminus U : U \subseteq N(u)}} x^{|U|}.$$

The degree of the neighborhood polynomial is given by the maximum degree of the graph. A vertex subset U is counted in the neighborhood polynomial if and only if in the complement of the graph at least one vertex exists, such that the intersection between the closed neighborhood of this vertex and the set U is empty.

Lemma 3.3. *Let* $\mathrm{N}(G,x)$ *be the neighborhood polynomial of the graph* G *and*

$$\overline{\mathrm{N}}(G,x) = \sum_{\substack{U \subseteq V \\ \exists u \in V \setminus U : \overline{U} \cap N[u] = \emptyset}} x^{|U|}.$$

Then

$$\mathrm{N}(G,x) = \overline{\mathrm{N}}(\overline{G},x).$$

The polynomial $\overline{\mathrm{N}}(G,x) = \mathrm{N}(\overline{G},x)$ counts the non-dominating sets in G and therefore the next theorem follows.

Theorem 3.4. *Let* $G = (V,E)$ *be a graph,* $\mathrm{D}(G,x)$ *be the domination polynomial of* G *and* $\mathrm{N}(\overline{G},x)$ *be the neighborhood polynomial of the complement of* G. *Then*

$$\mathrm{D}(G,x) + \mathrm{N}(\overline{G},x) = (1+x)^n.$$

Proof. The theorem follows directly from Lemma 3.3. $\qquad\qquad\qquad\qquad\square$

For the calculation of the domination polynomial in general graphs, there are two possible ways. The first way is to find recurrence equations and the second is to find representations of the polynomial which allow a (faster) computation of the polynomial. In this scope Kotek et al. [KPT13] introduced in 2013 the concept of the essential vertex subsets of a graph. A vertex subset $W \subseteq V$ is called *essential* if one vertex $u \in V$ exists with $N[u] \subseteq W$. Let $\mathrm{Ess}(G)$ be the set of all essential sets of the graph G. To calculate the domination polynomial it is enough to sum over these essential sets.

Theorem 3.5. *[KPT13] Let* $G = (V,E)$ *be a graph. Then*

$$\mathrm{D}(G,x) = (-1)^{|V|} \sum_{W \in \mathrm{Ess}(G)} (-1)^{|W|} \left((1+x)^{|\{v \in W | N[v] \subseteq W\}|} - 1 \right).$$

An open problem in the scope of the essential sets of a graph is to determine how many essential sets has a given graph. Or more precisely: Can the number of essential sets of a graph be bounded by some evaluations of the degree sequence, the minimum or maximum degree? E.g. the complete graph has exactly one essential set (the whole vertex set) and the complete bipartite graph $K_{m,n}$ has $2^n + 2^m - 3$ essential sets.

Remark 3.6. *The size of the smallest essential set equals* $\delta(G) + 1$.

As mentioned before, one interesting question about graph polynomials is: Does the graph polynomial fulfill some recurrence equations with respect to vertex or edge operations? Let $G = (V,E)$ be a graph and u be a vertex of G. Then $p_u(G)$ is the domination polynomial of $G - N[u]$ under the condition that all vertices in $N(u)$ are dominated in G. Together with this special domination polynomial Kotek et al. [Kot+12] proved the following theorem.

Theorem 3.7. *[Kot+12] Let* $G = (V,E)$ *be a graph and* $u \in V$. *Then*

$$\mathrm{D}(G,x) = \mathrm{D}(G-u,x) + x\,\mathrm{D}(G/u,x) + x\,\mathrm{D}(G-N[u],x) - (1+x)p_u(G).$$

Kotek et al. also showed in their paper that the domination polynomial does not satisfy any linear recurrence relation with the four vertex operations $-v$, $/v$, $-N[v]$ and $\setminus N[v]$. We can use the same method to extend their result with one additional operation, namely $\odot v$.

Theorem 3.8. *Let $G = (V, E)$ be an arbitrary graph and v be a vertex of this graph. Then for the domination polynomial there exists no linear recurrence equation with the operations $G-v$, G/v, $G - N[v]$, $G\backslash N[v]$ and $G \odot v$. More precisely, no rational functions $a, b, c, d, e \in \mathbb{R}(x)$ exist such that*

$$\begin{aligned}
\mathrm{D}(G, x) = {} & a\,\mathrm{D}(G - v, x) + b\,\mathrm{D}(G/v, x) + c\,\mathrm{D}(G - N[v], x) \\
& + d\,\mathrm{D}(G\backslash N[v], x) + e\,\mathrm{D}(G \odot v, x).
\end{aligned} \tag{3.1}$$

Proof. Suppose now that there exist rational functions a, b, c, d and e such that (3.1) is fulfilled. Let now the graphs G_i, for $i \in \{1, \ldots, 6\}$, be the K_2, K_3, P_3, K_4, P_4, and P_5, respectively. Now we apply the operations from the theorem to these graphs. In the complete graphs, the vertex v is an arbitrary vertex of the graph. In the case of the P_3 and the P_5 we choose the center vertex and in the case of P_4 we choose the second to last vertex. Then we obtain the Table 3.1.

	$G - v$	G/v	$G - N[v]$	$G\backslash N[v]$	$G \odot v$
K_2	K_1	K_1	$\emptyset$	K_1	K_1
K_3	K_2	K_2	$\emptyset$	K_1	$K_1 \cup K_1$
P_3	$K_1 \cup K_1$	K_2	$\emptyset$	K_1	$K_1 \cup K_1$
K_4	K_3	K_3	$\emptyset$	K_1	$K_1 \cup K_1 \cup K_1$
P_4	$K_2 \cup K_1$	P_3	K_1	K_2	$K_2 \cup K_1$
P_5	$K_2 \cup K_2$	P_4	$K_1 \cup K_1$	P_3	$K_2 \cup K_2$

Tab. 3.1: Simple graphs and some vertex operations.

For all graphs of the Table 3.1 we can calculate the domination polynomials and obtain Table 3.2.

	$G - v$	G/v	$G - N[v]$	$G\backslash N[v]$	$G \odot v$
$x^2 + 2x$	x	x	1	x	x
$(1 + x)^3 - 1$	$x^2 + 2x$	$x^2 + 2x$	1	x	x^2
$x^3 + 3x^2 + x$	x^2	$x^2 + 2x$	1	x	x^2
$(1 + x)^4 - 1$	$(1 + x)^3 - 1$	$(1 + x)^3 - 1$	1	x	x^3
$x^2(x + 2)^2$	$x^3 + 2x^2$	$x^3 + 3x^2 + x$	x	$x^2 + 2x$	$x^3 + 2x^2$
$x^5 + 5x^4 + 8x^3 + 3x^2$	$(x^2 + 2x)^2$	$x^2(x + 2)^2$	x^2	$x^3 + 3x^2 + x$	$(x^2 + 2x)^2$

Tab. 3.2: Domination polynomials of some simple graphs.

The domination polynomials in Table 3.2 give us a system of linear equations. This system has no rational solution for the variables a, b, c, d and e (this can be easily proved using a computer algebra system), which is a contradiction to the assumption and therefore the theorem is proved. $\qquad\square$

Kotek et al. [Kot+12] also showed that for the domination polynomial no recurrence equation exists with the deletion, contraction and extraction of an edge. But it is possible to prove a theorem which uses a combination of vertex and edge operations, as follows.

Theorem 3.9. *[Kot+12] Let $G = (V, E)$ be a graph and $e = \{u, v\} \in E$ be an edge of G. Then*

$$
\begin{aligned}
\mathrm{D}(G, x) = {}& \mathrm{D}(G - e, x) + \frac{x}{x-1} \Big[\mathrm{D}(G - e/u, x) + \mathrm{D}(G - e/v, x) \\
& - \mathrm{D}(G/u, x) - \mathrm{D}(G/v, x) - \mathrm{D}(G - N[u], x) - \mathrm{D}(G - N[v], x) \\
& + \mathrm{D}(G - e - N[u], x) + \mathrm{D}(G - e - N[v], x) \Big].
\end{aligned}
$$

3.1 Graph Products

In literature we find a lot of papers about the domination number of product graphs (e.g. see [Ala+11; Klo99a; Klo99b]), but almost no attention has been given to the domination polynomial of graph products. Kotek et al. [KPT14] has investigated the domination polynomial of Cartesian products. In this section we prove some results about the lexicographic product. In most cases it is not possible to give formulas for graph products with two arbitrary graphs and therefore we look for results for the case that some special graphs, e.g. the complete graph, are involved.

Theorem 3.10. *Let $G = (V, E)$ be a graph with at least two vertices. Then the domination polynomial of the lexicographic product of the graph G and the complete graph K_n ($n \geq 2$) can be calculated with*

$$
\mathrm{D}(G \cdot K_n, x) = \mathrm{D}(G, (1+x)^n - 1).
$$

Proof. A vertex (v, w) of the product graph is adjacent to all vertices in the same row and to all vertices in the row R_u if $u \in N_G(v)$. Hence, the theorem follows. $\qquad\square$

Theorem 3.11. *Let $G = (V, E)$ be a connected graph with m vertices and $m, n \geq 2$. Then*

$$
\mathrm{D}(K_n \cdot G, x) = (1+x)^{nm} - n(\overline{N}(G, x) - 1) - 1.
$$

Proof. A vertex (v, w) of the product graph is adjacent to all vertices outside the row R_v and to some vertices of the row R_v depending on the neighborhood of $w \in V(G)$ in G. Therefore, every non-empty vertex subset of the product graph is a dominating set except of subsets that only consist of vertices of one row and these vertices do not correspond to a dominating set in G. These sets are counted by $\overline{N}(G, x) - 1$ and the theorem follows. $\qquad\square$

Theorem 3.12. *Let $G = (V, E)$ be a graph with m vertices and $m, n \geq 2$. Then the domination polynomial of the lexicographic product of the path P_n with the graph G can be calculated with*

$$
\mathrm{D}(P_n \cdot G, x) = \mathrm{D}(G, x) f_{n-1} + \left(\overline{N}(G, x) - 1 \right) \left((1+x)^m - 1 \right) f_{n-2} + g_{n-1}.
$$

f_n denotes the domination polynomial of the graph $P_n \cdot G$ under the condition that the first row is already dominated and g_n denotes the domination polynomial of $P_n \cdot G$ under the condition that in the first row at least one vertex is dominating.

Proof. Again, we can distinguish three possible cases with respect to the dominating vertices in the first row: (1) The dominating vertices of the first row correspond to a dominating set in G, (2) at least one vertex in the first row is dominating, but the dominating vertices in the first row do not correspond to a dominating set in G and (3) no vertex in the first row is dominating.

1. If in the first row at least one vertex is dominating, then in the second row of the product graph all vertices are dominated. The number of dominating sets of the first row is counted by $\mathrm{D}(G, x)$ and the number of dominating sets of the remaining graph with f_{n-1}.

2. The number of non-dominating sets in the first row is counted by $\overline{N}(G, x) - 1$. To dominate the non-dominated vertices in the first row, there must be at least one vertex in the second row dominating and therefore all vertices in the third row are dominated.

3. If in the first row no vertex is dominating, then in the second row at least one vertex must be dominating.

The sum of the polynomials of the three cases yields the theorem. $\square$

Lemma 3.13. *In compliance with the requirements of the previous theorem for the polynomials f_n (for $n \geq 2$) and g_n (for $n \geq 3$) the following equations are valid:*

$$f_n = \mathrm{D}(P_{n-1} \cdot G, x) + ((1+x)^m - 1) f_{n-1},$$
$$g_n = \mathrm{D}(G, x)f_{n-1} + \left(\overline{N}(G, x) - 1\right)((1+x)^m - 1) f_{n-2}.$$

The initial conditions are

$$f_1 = (1+x)^m,$$
$$g_1 = \mathrm{D}(G, x),$$
$$g_2 = \mathrm{D}(G, x)(1+x)^m + \left(\overline{N}(G, x) - 1\right)((1+x)^m - 1).$$

Proof. Analog to the proof of Theorem 3.12. $\square$

Theorem 3.14. *Let $G = (V, E)$ be a graph with m vertices, $m \geq 2$. Then the domination polynomial of the lexicographic product of the cycle C_n ($n \geq 4$) with the graph G can be calculated with*

$$\mathrm{D}(C_n \cdot G, x) = \mathrm{D}(G, x)h_{n-1} + l_{n-1}$$
$$+ \left(\overline{N}(G, x) - 1\right)\left[2((1+x)^m - 1)f_{n-3} + ((1+x)^m - 1)^2 h_{n-3}\right].$$

h_n denotes the domination polynomial of the graph $P_n \cdot G$ under the condition that the first and the last row is already dominated. Furthermore, l_n denotes the domination polynomial of the graph $P_n \cdot G$ under the condition that in the first row at least one vertex or in the last row at least one vertex is dominating.

Proof. Analog to the proof of Theorem 3.12. $\square$

Lemma 3.15. *Let h_n and l_n be the polynomials defined in the last theorem, with $n \geq 2$. Furthermore, i_n, for $n \geq 5$, denotes the domination polynomial of the graph $P_n \cdot G$ under the condition that at least one vertex in the first row and at least one vertex in the last row is dominating. Then*

$$h_n = ((1+x)^m - 1)h_{n-1} + f_{n-1},$$
$$l_n = 2i_{n-1} + i_n,$$
$$i_n = \mathrm{D}(G, x)^2 h_{n-2} + 2\left(\overline{N}(G, x) - 1\right)\mathrm{D}(G, x)((1+x)^m - 1)h_{n-3}$$
$$+ \left(\overline{N}(G, x) - 1\right)^2 ((1+x)^m - 1)^2 h_{n-4}.$$

The initial conditions are

$$
\begin{aligned}
h_1 &= (1+x)^m, \\
i_1 &= \mathrm{D}(G,x),\ i_2 = ((1+x)^m - 1)^2, \\
i_3 &= \mathrm{D}(G,x)^2 h_1 + \left(2\left(\overline{N}(G,x) - 1\right)\mathrm{D}(G,x) + \left(\overline{N}(G,x) - 1\right)^2\right)((1+x)^m - 1) \ \text{ and} \\
i_4 &= \mathrm{D}(G,x)^2 h_2 + 2\left(\overline{N}(G,x) - 1\right)\mathrm{D}(G,x)((1+x)^m - 1)h_1 \\
&\quad + \left(\overline{N}(G,x) - 1\right)^2 ((1+x)^m - 1)^2.
\end{aligned}
$$

Proof. The proof of the recurrence equations for h_n and i_n is analog to the proof of Theorem 3.12.

The polynomial l_n is the domination polynomial of the graph $P_n \cdot G$ under the condition that in the first row or in the last row at least one vertex is dominating. Therefore, we have two cases: (1) Exactly in one of the two rows dominating vertices exist and (2) in both of the two rows dominating vertices exist. The last case is counted by i_n. Suppose that in the first row at least one vertex is dominating and in the last row no vertex is dominating. To dominate the vertices in the last row, at least one vertex in the second to last row must be dominating. This is counted with i_{n-1}. The sum of the two cases yields the theorem. □

3.2 Special Graph Classes

In this section we investigate the domination polynomial of some special graph classes. For several graph classes, results are published in the literature (e.g. see [AO09; AT10; DT12]).

3.2.1 Complete and Nearly Complete Graphs

If in a complete graph at least one vertex is dominating, then all other vertices are dominated. This yields immediately the dominating polynomial of the complete graph

$$\mathrm{D}(K_n, x) = (1+x)^n - 1.$$

Theorem 3.16. *Let $K_n^k = (V, E)$ be a simple k-bounded complete graph with n vertices of the type $\Lambda(K_n^k) = [n_0, n_1, n_2, \ldots, n_l]$. Then*

$$
\begin{aligned}
\mathrm{D}(K_n^k, x) =\,&((1+x)^{n_0} - 1)(1+x)^{n - n_0} \\
&+ \sum_{i=1}^{l-1}\left[((1+x)^{n_i} - 1)\prod_{j=i+1}^{l}(1+x)^{n_j} - ((1+x)^{n_i} - 1 - x^{n_i})\right] + x^{n_l}.
\end{aligned}
$$

Proof. If at least one vertex of the clique of size n_0 is dominating, then the remaining $n - n_0$ vertices are dominated. This leads to the first part of the equation.

Let $i \in \{1, \ldots, l - 1\}$, the vertices in $V_0, \ldots, V_{i-1}$ are non-dominating and at least one and at most $n_i - 1$ vertices are dominating in V_i. Then these vertices dominate all vertices in $V \setminus V_i$, but the non-dominating vertices in V_i will not be dominated. Therefore, at least one vertex in $V_{i+1}, \ldots, V_l$ has to be dominating. This is counted by

$$((1+x)^{n_i} - 1 - x^{n_i})\left(\prod_{j=i+1}^{l}(1+x)^{n_j} - 1\right).$$

If in V_i all vertices are dominating, then all vertices are dominated and the vertices in the vertex set $V \setminus V_i$ can either be dominating or not. This yields

$$x^{n_i} \prod_{j=i+1}^{l} (1+x)^{n_j} + x^{n_l}$$

and therefore the theorem follows. $\qquad \square$

Theorem 3.17. *Let $M \subset E$ be a perfect matching of the complete graph $K_n = (V, E)$. Then*
$$\mathrm{D}(K_n - M, x) = (1+x)^n - 1 - nx.$$

Proof. All vertex subsets of size greater or equal two are dominating sets. $\qquad \square$

Corollary 3.18. *Let $M \subset E$ be a matching of the complete graph $K_n = (V, E)$ and $m = |M|$. Then*
$$\mathrm{D}(K_n - M, x) = (1+x)^{n-2m}(1+x)^{2m} - 2mx - 1.$$

Theorem 3.19. *Let $G = (V, E)$ be a complete graph with k holes. Let n_i be the size of the i-th hole in the graph and $m_j = \sum_{i=1}^{j} n_i$, for $j \in \{1, \ldots, k\}$. Then*

$$\mathrm{D}(G, x) = (1+x)^{n-m_k} + \sum_{i=1}^{k} \left[\left((1+x)^{n-m_i} - 1 \right) \left((1+x)^{n_i} - 1 \right) + \mathrm{D}(C_{n_i}, x) \right] - 1.$$

Proof. Apply Theorem 3.1 iteratively to the holes and the rest of the graph. $\qquad \square$

Theorem 3.20. *Let $G = (V, E)$ be a (n, k)-star. Then*
$$\mathrm{D}(S_{n,k}, x) = ((1+x)^k - 1)(1+x)^{n-k} + x^{n-k}.$$

Proof. If at least one vertex in the start clique is dominating, then all other vertices are dominated. This is counted by $((1+x)^k - 1)(1+x)^{n-k}$. If all vertices in the start clique are non-dominating, then every leaf-node must be dominating and therefore the theorem follows. $\qquad \square$

3.2.2 Bipartite and Nearly Bipartite Graphs

Theorem 3.21. *[AP09b] Let $K_{n,m} = (X \cup Y, E)$ be a complete bipartite graph with $|X| = n$ and $|Y| = m$. Then*
$$\mathrm{D}(K_{n,m}, x) = ((1+x)^n - 1)((1+x)^m - 1) + x^m + x^n.$$

Theorem 3.22. *Let G be the graph obtained from a complete bipartite graph $K_{n,n} = (X \cup Y, E)$ by removing all edges of a perfect matching M. Then*

$$\mathrm{D}(G, x) = nx^2(1+x)^{n-1}$$
$$+ \sum_{i=2}^{n} \binom{n}{i} x^i \left[\left((1+x)^i - 1 \right) (1+x)^{n-i} + \left(1+x \right)^{n-i} - 1 - (n-i)x \right].$$

Proof. If in X exactly one vertex is dominating, then all but one vertices in Y are dominated. Hence, this single vertex must also be dominating. This is counted by $nx^2(1+x)^{n-1}$.

Let now W be the set of dominating vertices of X and let $|W| = i$, for $i \in \{2, \ldots, n\}$. Furthermore, let $U = \{u \in Y : \exists w \in W : \{u, w\} \notin E(G)\}$. The vertices in W dominate all vertices in Y. To dominate the $n - i$ non-dominated vertices in X we have two possibilities: At least one vertex in U is dominating or all vertices in U are non-dominating and in $Y \setminus U$ at least two vertices are dominating. This yields the sum in the theorem. $\qquad \square$

3.2.3 Paths and k-Paths

The following theorem gives a recurrence equation for the path P_n. This result can also be obtained from Theorem 3.7.

Theorem 3.23. *[AP09a] Let P_n be the path with $n \geq 4$ vertices. Then*

$$\mathrm{D}(P_n, x) = x(\mathrm{D}(P_{n-1}, x) + \mathrm{D}(P_{n-2}, x) + \mathrm{D}(P_{n-3}, x)),$$

with the initial conditions

$$\begin{aligned}
\mathrm{D}(P_1, x) &= x, \\
\mathrm{D}(P_2, x) &= x^2 + 2x, \\
\mathrm{D}(P_3, x) &= x^3 + 3x^2 + x.
\end{aligned}$$

Proof. Let v be the second vertex of the path. Applying Theorem 3.7 we obtain

$$\begin{aligned}
\mathrm{D}(P_n, x) &= \mathrm{D}(P_n - v, x) + x\,\mathrm{D}(P_n/v, x) + x\,\mathrm{D}(P_n - N[v], x) - (1+x)p_u(P_n) \\
&= x\,\mathrm{D}(P_{n-2}, x) + x\,\mathrm{D}(P_{n-1}, x) + x\,\mathrm{D}(P_{n-3}, x).
\end{aligned}$$

$\square$

We can generalize the recurrence equation for the path to the simple k-path.

Theorem 3.24. *Let $P_n^{(k)}$ be a simple k-path with n vertices ($k \geq 1$ and $n \geq k + 2$). Then*

$$\mathrm{D}(P_n^{(k)}, x) = x \sum_{i=1}^{2k+1} \mathrm{D}(P_{n-i}^{(k)}, x),$$

with the initial conditions

$$\begin{aligned}
\mathrm{D}(P_i^{(k)}, x) &= 1, \text{ for } i \leq 0, \\
\mathrm{D}(P_i^{(k)}, x) &= (1+x)^i - 1, \text{ for } i \in \{1, \ldots k+1\}.
\end{aligned}$$

Proof. The proof uses the same idea as the proof of Theorem 3.23. $\square$

Let $\mathrm{D}(P_n^{(k)}, x)$ be the domination polynomial of the simple k-path $P_n^{(k)}$. Then $2^n\,\mathrm{D}(P_n^{(k)}, 1)$ yields the number of 01-words with length n, which contain no $2k + 1$ zeros in a row and no $k + 1$ leading and $k + 1$ trailing zeros.

3.2.4 Cycles and k-Cycles

Theorem 3.25. *[DT12] Let C_n be a cycle with $n \geq 4$ vertices. Then*

$$\mathrm{D}(C_n, x) = x(\mathrm{D}(C_{n-1}, x) + \mathrm{D}(C_{n-2}, x) + \mathrm{D}(C_{n-3}, x)),$$

with the initial conditions $\mathrm{D}(C_1, x) = x$, $\mathrm{D}(C_2, x) = x^2 + 2x$ und $\mathrm{D}(C_3, x) = x^3 + 3x^2 + 3x$.

Lemma 3.26. *Let W_n be a wheel with $n \geq 4$ vertices. Then*

$$\mathrm{D}(W_n, x) = \mathrm{D}(C_{n-1}, x) + x(1+x)^{n-1}.$$

Proof. If the center vertex of the wheel is dominating, then all other vertices are dominated and they can either be dominating or not. This will be counted by $x(1 + x)^{n-1}$. If the center vertex is non-dominating, then the dominating vertices of the cycle must form a dominating vertex set. □

Theorem 3.27. *Let C_n be a cycle with at least five vertices. Then*

$$\mathrm{D}(\overline{C_n}, x) = (1 + x)^n - 1 - nx - nx^2.$$

Proof. Every vertex subset of size at least three is a dominating vertex set. A vertex subset of size two is non-dominating if the two vertices are adjacent and have a common non-adjacent vertex. In the $\overline{C_n}$ we have n possibilities to choose such a vertex subset. Hence, the theorem follows. □

We can generalize the recurrence equation for the cycle to the k-cycle.

Theorem 3.28. *Let $C_n^{(k)}$ be a k-cycle with n vertices ($k \geq 1$ and $n \geq 2k + 2$). Then*

$$\mathrm{D}(C_n^{(k)}, x) = x \sum_{i=1}^{2k+1} \mathrm{D}(C_{n-i}^{(k)}, x),$$

with the initial condition

$$\mathrm{D}(C_i^{(k)}, x) = (1 + x)^i - 1, \text{ for } i \in \{1, \ldots 2k + 1\}.$$

Proof. The proof uses the same idea as the proof of Theorem 3.25 (see [DT12]). □

Let $\mathrm{D}(C_n^{(k)}, x)$ be the domination polynomial of the k-cycle. Then $2^n \mathrm{D}(C_n^{(k)}, 1)$ yields the number of cyclic 01-words of length $n \geq 2k + 1$, which contains no subword with $2k + 1$ consecutive zeros. For these numbers W. Moser proved following theorem.

Theorem 3.29. *[Mos93] Let $L_w(n)$ be the number of cyclic words consisting of zeros and ones, which contains no $w + 1$ zeros in series. Then*

$$L_w(n) = \begin{cases} 2^n, & \text{if } n \in \{1, \ldots, w\} \\ 2^n - 1, & \text{if } n = w + 1 \\ L_w(n-1) + \cdots + L_w(n-1-w) + n - 2(w+1), & \text{if } w + 2 \leq n \leq 2w + 1 \\ L_w(n-1) + \cdots + L_w(n-1-w), & \text{if } n \geq 2w + 2. \end{cases}$$

Corollary 3.30. *Let $\mathrm{D}(C_n^{(2)}, x)$ be the domination polynomial of the 2-cycle. Then $2^n \mathrm{D}(C_n^{(2)}, 1)$ yields the n_{th} pentanacci number with initial conditions $a(0) = 5$, $a(1) = 1$, $a(2) = 3$, $a(3) = 7$, $a(4) = 15$ (series A074048 of "The On-Line Encyclopedia of Integer Sequences" [OEI14]).*

3.2.5 Trees

The calculation of many graph polynomials can be done in polynomial time when restricted to the class of trees. In this section, first we present a general algorithmic approach to calculate graph polynomials in trees and then we specify it for the calculation of the domination polynomial. Let $T = (V, E)$ be a tree and $v \in V$ be an arbitrary vertex of the tree. Let T_v be the rooted tree obtained from T with the root v and let T_u^v be the rooted subtree (with

respect to the root v) with the root u of the subtree and all descendants of u. Let now P_u be a vector with n graph polynomials for the subtree T_u^v as elements. These graph polynomials have some special properties with respect to the vertex u. The idea is to start the calculation beginning with the leaves of the tree T_v and going upwards in the tree until we reach the root. For these upward steps, we need two operations: The first is the $\oplus$-operation. By this operation we are going one step upwards in the tree and adding the father w of the vertex u and the edge $\{u, w\}$. The second operation is the $\otimes$-operation. By this operation we merge two P-vectors of the same vertex. In Figure 3.1 we see a tree with the root 4 and the corresponding calculation steps. In the algorithm, the vector of the root v will be calculated in the last step. This gives the graph polynomial for the whole tree.
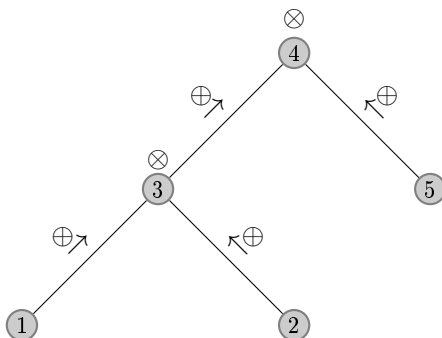
Fig. 3.1: Rooted tree with root 4 and the corresponding calculation steps.

To prove the correctness of such an algorithm, it is necessary to show that the initial assignments to the leaves, the two operations and the final calculation of the polynomial in the root of the tree are correct. The running time of such an algorithm depends mainly on the two operations $\oplus$ and $\otimes$. Let $O(\oplus)$ and $O(\otimes)$ be the complexity of the two operations. Then the complexity of the whole algorithm is $O(mO(\oplus) + nO(\otimes))$.

Now we specify the operations to calculate the domination polynomial. We assign to every vertex u of the tree a vector with three components. We denote the i-th component of P_u with P_u^i. The first component P_u^1 yields the domination polynomial of the subtree T_u^v under the condition that the vertex u is dominating, the second component P_u^2 under the condition that the vertex u is non-dominating, but it will be dominated from at least one son. The third component P_u^3 yields the domination polynomial under the condition that u and all its sons are non-dominating. Additionally, this polynomial counts non-dominating sets in the subtree T_u^v, which are dominated sets in $T_u^v - u$. Let $v \in V$ be the root and $u \in V$ be a leaf of the tree. We assign to u the vector

$$P_u = \begin{pmatrix} x \\ 0 \\ 1 \end{pmatrix}.$$

If we go one step upwards in the tree, the new vertex w can either be dominating or not. If it is dominating, then it dominates its son. If it is non-dominating, then only if the son is dominating or it is already dominated, we can obtain a dominating set. This yields

$$P_u \oplus w = P_w = \begin{pmatrix} x(P_u^1 + P_u^2 + P_u^3) \\ P_u^1 \\ P_u^2 \end{pmatrix}.$$

Let now w be a vertex of the tree with two sons and P_w and Q_w be the two vectors calculated for w from its sons. Now we merge these two vectors

$$P_w \otimes Q_w = \begin{pmatrix} P_w^1(Q_w^1/x + Q_w^3) + Q_w^1 P_w^3 \\ P_w^2 Q_w^2 + P_w^2 Q_w^3 + P_w^3 Q_w^2 \\ P_w^3 Q_w^3 \end{pmatrix}.$$

The sum of the first two components of the vector of the root yields the domination polynomial of the whole tree.

3.3 The Domination Reliability Polynomial

Now we want to look at the domination problem from the reliability point of view. We are interested in the probability that in a graph with random failing vertices or edges, a dominating set (with some properties) exists. In this section we only show some possible directions for further research. Assume that the vertices of the graph are dominating with a given probability p and the edges are perfectly reliable.

Let $G = (V, E)$ be a graph whose vertices fail randomly and independently with a given probability q_v, for all $v \in V$. A failure in the context of domination means, that the vertex is not in the dominating set. In such a graph we are interested in the reliability that a dominating set exists. If we assume that $q = q_v$, for all $v \in V$, and $p = 1 - q$, then we can define the *domination reliability polynomial* as

$$\mathrm{DRel}(G, p) = (1 - p)^{|V|} \sum_{\substack{W \subseteq V \\ N[W] = V}} \left(\frac{p}{1 - p} \right)^{|W|}.$$

The domination reliability polynomial was first introduced by Dohmen and Tittmann [DT12]. They also showed that the domination polynomial and the domination reliability polynomial are equivalent.

Theorem 3.31. *[DT12] Let $G = (V, E)$ be a graph whose vertices fail randomly and independently with equal probability $q = 1 - p$. Then*

$$\mathrm{D}(G, x) = (1 + x)^{|V|} \mathrm{DRel}\left(G, \frac{x}{1 + x} \right)$$

and

$$\mathrm{DRel}(G, p) = (1 - p)^{|V|} \mathrm{D}\left(G, \frac{p}{1 - p} \right).$$

For more results and properties of the domination reliability polynomial we refer the interested reader to the paper of Dohmen and Tittmann [DT12].

Assume now that the edges of the graph are subject to random failure and the vertices are perfectly reliable. First we ask for the probability that a given vertex subset is dominating.

Definition 3.32. *Let $G = (V, E)$ be a graph. Suppose now that the edges fail randomly and independently with the probability $q = 1 - p$. Then the* edge failure domination reliability polynomial $\mathrm{EDRel}(G, A, p)$ *is the probability that the vertex subset $A \subseteq V$ is dominating in G:*

$$\mathrm{EDRel}(G, A, p) = (1 - p)^{|E|} \sum_{\substack{F \subseteq E \\ N_{G\langle F \rangle}[A] = V}} \left(\frac{p}{1 - p} \right)^{|F|}.$$

It follows directly from the definition that only the edges in δA are necessary for the calculation of $\mathrm{EDRel}(G, A, p)$. Hence, the edge failure domination reliability equals zero if at least one vertex exists which is not in the neighborhood of A.

Theorem 3.33. *Let $G = (V, E)$ be a graph and $A \subseteq V$. Then*

$$\mathrm{EDRel}(G, A, p) = \prod_{v \in V \setminus A} (1 - q^{|N_G(v) \cap A|}).$$

Proof. From every vertex in $V \setminus A$ at least one edge to a vertex in A must be intact. The term $1 - q^{|N_G(v) \cap A|}$ yields this probability for one vertex $v \in V \setminus A$. $\qquad \square$

Theorem 3.33 shows that the edge failure domination reliability can be calculated in polynomial time with respect to the number of vertices of the graph.

In the context of domination problems in graphs with failing edges, a lot of interesting questions exist. So, one may ask how is the probability that in a graph with random failing edges a dominating set with at most k, for $1 \leq k \leq n - 1$, vertices exist? One can also add some restrictions on the dominating sets, e.g. they must be *perfectly dominating* (every dominated vertex is dominated from exactly one vertex). It is also possible to relax the domination condition and introduce d-dominating vertex sets (a vertex subset $W \subseteq V$ is called d-*dominating* if every vertex in $V \setminus W$ has a distance of at most d from a vertex in W).

4 The Independent Domination Polynomial

A vertex subset W of a graph $G = (V, E)$ is called independent dominating if $N[W] = V$ and $|W| = \text{iso}(G[W])$. The independent dominating number $\gamma_i(G)$ is the minimum size of an independent dominating set of the graph G. A survey of recent results of the independent domination number is given by Goddard and Henning in [GH13]. In this chapter we investigate the independent domination polynomial, which is the ordinary generating function for the number of independent dominating sets in a graph.

Definition 4.1. *Let $G = (V, E)$ be a simple graph and $d_k^i(G)$ be the number of independent dominating sets of size k in G. Then the* independent domination polynomial *is defined as*

$$\mathrm{D}_i(G, x) = \sum_{k=1}^{n} d_k^i(G) x^k.$$

The independent domination polynomial $\mathrm{D}_i(G, x)$ can be obtained from the trivariate domination polynomial $\mathrm{Y}(G; x, y, z)$ (see Equation (5.7) on page 79). Like many other graph polynomials, the independent domination polynomial is multiplicative with respect to the components of the graph (which also follows from the connection to the trivariate domination polynomial).

Lemma 4.2. *Let $G = (V, E)$ be a graph with two components $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Then*
$$\mathrm{D}_i(G, x) = \mathrm{D}_i(G_1, x) \, \mathrm{D}_i(G_2, x).$$

Proof. The proof of the lemma follows directly from the definition of the polynomial. $\square$

Theorem 4.3. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be graphs. Then we obtain the following equation for the join of the two graphs.*

$$\mathrm{D}_i(G * H, x) = \mathrm{D}_i(G, x) + \mathrm{D}_i(H, x).$$

Proof. Every independent set in G dominates all vertices in H and vice versa. If $S \subseteq V(G)$ is independent set but not a dominating set, then all vertices in H are adjacent to the vertices in S and therefore the set S cannot be extended to an independent dominating set with vertices of H. Hence, the theorem follows. $\square$

Theorem 4.4. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be graphs and $n = |V(G)|$. Then*
$$\mathrm{D}_i(G \circ H, x) = \mathrm{D}_i(H, x)^n \, \mathrm{I}\left(G, \frac{x}{\mathrm{D}_i(H, x)}\right).$$

Proof. Every independent vertex subset of G can be extended to an independent dominating set in $G \circ H$. Let $S \subseteq V(G)$ be an independent set in G, $|S| = k$ and $H_1, \ldots, H_{n-k}$ the $n - k$ copies of H which are non-adjacent to vertices of S in $G \circ H$. Then for all arbitrary

independent dominating sets $S_i \subseteq V(H_i)$, $i \in \{1, \ldots, n-k\}$, the set $S \cup S_1 \cup \cdots \cup S_{n-k}$ is an independent dominating set in $G \circ H$.

Let i_k be the coefficient of x^k in $\mathrm{I}(G, x)$. Then

$$\sum_{k=0}^{n} i_k x^k \, \mathrm{D}_i(H, x)^{n-k} = \mathrm{D}_i(H, x)^n \sum_{k=0}^{n} i_k x^k \, \mathrm{D}_i(H, x)^{-k}$$

and therefore the theorem follows. $\qquad\square$

Corollary 4.5. *Let $G = (V, E)$ be a graph and E_r be an edgeless graph with r vertices. Then*

$$\mathrm{D}_i(G \circ E_r, x) = x^{rn} \, \mathrm{I}(G, x^{1-r}).$$

It is also possible to prove a nice formula for the r-expansion of a graph.

Definition 4.6. *[GH13] Let $G = (V, E)$ be a graph. Then the r-expansion $\exp(G, r)$ is the graph obtained from G by replacing every vertex $v \in V$ with an independent set I_v of size r and replacing every edge $\{u, v\} \in E$ with a complete bipartite graph with the bipartite sets I_u and I_v.*

Theorem 4.7. *Let $G = (V, E)$ be a graph and $\exp(G, r)$ be the r-expansion of it. Then*

$$\mathrm{D}_i(\exp(G, r), x) = \mathrm{D}_i(G, x^r).$$

Proof. Let $W \subseteq V$ be an independent dominating set in G. Then in $\exp(G, r)$, all r vertices in I_w, for $w \in W$, must be dominating and all vertices in I_u, for $u \in V \backslash W$, are non-dominating because of the complete bipartite graphs between the vertices in I_w and I_u. Hence, every independent dominating set in G can be expanded to exactly one independent dominating set in $\exp(G, r)$ and vice versa. $\qquad\square$

Theorem 4.8. *Let $G = (V, E)$ be a connected graph with at least two vertices. Then*

$$\sum_{W \subseteq V} (-1)^{|W|} \, \mathrm{D}_i(G[W], x) = 1.$$

Proof. The proof follows the proof in [KPT13] with some minor changes. First of all, we insert the definition of the independent domination polynomial in the equation and change the order of the summation.

$$\sum_{W \subseteq V} (-1)^{|W|} \, \mathrm{D}_i(G[W], x) = \sum_{W \subseteq V} (-1)^{|W|} \sum_{\substack{U \subseteq W \\ N_{G[W]}[U] = W \\ \mathrm{iso}(G[U]) = |U|}} x^{|U|}$$

$$= \sum_{\substack{U \subseteq V \\ \mathrm{iso}(G[U]) = |U|}} x^{|U|} \sum_{\substack{W : U \subseteq W \\ N_{G[W]}[U] = W}} (-1)^{|W|}$$

$$= \sum_{\substack{U \subseteq V \\ \mathrm{iso}(G[U]) = |U|}} x^{|U|} \sum_{W : U \subseteq W \subseteq N_{G[W]}[U]} (-1)^{|W|} \qquad (4.1)$$

$$= \sum_{\substack{U \subseteq V \\ \mathrm{iso}(G[U]) = |U|}} x^{|U|} \sum_{W : U \subseteq W \subseteq N_G[U]} (-1)^{|W|} \qquad (4.2)$$

$$= \sum_{\substack{U \subseteq V \\ \text{iso}(G[\overline{U}])=|U|}} (-x)^{|U|} \sum_{Y \subseteq N_G(U)} (-1)^{|Y|}$$

$$= 1.$$

In Equation (4.1) we sum over all vertex subsets W such that U is an independent dominating set in $G[W]$. The condition $W \subseteq N_{G[W]}[U]$ in Equation (4.1) guarantees that we sum only over subsets W such that U is a dominating set in $G[W]$. Hence, in the inner sum W can be every subset from $N_G[U]$. With these considerations we obtain Equation (4.2). Because of the fact that U is included in every subset W of the inner sum, the summation is performed only over vertex subsets included in $N_G(U)$ and $(-1)^{|U|}$ is factored out from the inner sum. The second sum vanishes for every set U which is not equal V or $\emptyset$ and therefore we obtain the theorem.

$\square$

Remark 4.9. *Let $G = (\{v\}, \emptyset)$ be a graph with one vertex. Then*

$$\sum_{W \subseteq V} (-1)^{|W|} \, \mathrm{D}_i(G[W], x) = 1 - x.$$

If an arbitrary graph has more than one component we obtain the following corollary as a consequence of Theorem 4.8 and Remark 4.9.

Corollary 4.10. *Let $G = (V, E)$ be a graph. Then*

$$\sum_{W \subseteq V} (-1)^{|W|} \, \mathrm{D}_i(G[W], x) = (1 - x)^{\text{iso}(G)}.$$

Applying Möbius inversion to Corollary 4.10 yields the next corollary.

Corollary 4.11. *Let $G = (V, E)$ be a graph. Then*

$$\mathrm{D}_i(G, x) = \sum_{W \subseteq V} (-1)^{|W|} (1 - x)^{\text{iso}(G[W])}.$$

The previous corollary yields a formula to calculate the coefficients of the independent domination polynomial.

Corollary 4.12. *Let $G = (V, E)$ be a graph with n vertices. Then*

$$\mathrm{D}_i(G, x) = \sum_{k=0}^{n} x^k \sum_{\substack{W \subseteq V \\ \text{iso}(G[\overline{W}]) \geq k}} (-1)^{|W|+k} \binom{\text{iso}(G[W])}{k}.$$

Proof. Using the Corollary 4.11, we obtain

$$\mathrm{D}_i(G, x) = \sum_{W \subseteq V} (-1)^{|W|} (1 - x)^{\text{iso}(G[W])}$$

$$= \sum_{W \subseteq V} (-1)^{|W|} \sum_{k=0}^{\text{iso}(G[W])} \binom{\text{iso}(G[W])}{k} (-x)^k$$

$$= \sum_{k=0}^{n} (-x)^k \sum_{W \subseteq V} (-1)^{|W|} \binom{\mathrm{iso}(G[W])}{k}$$

$$= \sum_{k=0}^{n} x^k \sum_{\substack{W \subseteq V \\ \mathrm{iso}(G[W]) \geq k}} (-1)^{|W|+k} \binom{\mathrm{iso}(G[W])}{k}.$$

$\square$

Definition 4.13. *Let $G = (V, E)$ be a graph and W be a vertex subset of the graph. The set W is called* i-essential *if W contains the open neighborhood of at least one vertex of $V \setminus W$. We denote the family of i-essential sets of G by $\mathrm{Ess}_i(G)$:*

$$\mathrm{Ess}_i(G) = \{X \subseteq V : \exists v \in V \setminus X : N(v) \subseteq X\}.$$

Theorem 4.14. *Let $G = (V, E)$ be a graph with n vertices. Then*

$$\mathrm{D}_i(G, x) = (-1)^n \sum_{U \subseteq \mathrm{Ess}_i(G)} (-1)^{|U|} \left((1-x)^{|\{v \in V \setminus U | N_G(v) \subseteq U\}|} - 1 \right).$$

Proof. Using complements with respect to V in the sum of Corollary 4.11 yields:

$$\mathrm{D}_i(G, x) = \sum_{W \subseteq V} (-1)^{|W|} (1-x)^{\mathrm{iso}(G[W])}$$

$$= \sum_{U \subseteq V} (-1)^{|V \setminus U|} (1-x)^{\mathrm{iso}(G[V \setminus U])}$$

$$= (-1)^n \sum_{U \subseteq V} (-1)^{|U|} (1-x)^{|\{v \in V \setminus U | N_G(v) \subseteq U\}|}$$

$$= (-1)^n \left(\sum_{U \subseteq \mathrm{Ess}_i(G)} (-1)^{|U|} (1-x)^{|\{v \in V \setminus U | N_G(v) \subseteq U\}|} + \sum_{\substack{U \subseteq V \\ |\{v \in V \setminus U | N_G(v) \subseteq U\}| = 0}} (-1)^{|U|} \right).$$

The second sum equals a constant term and therefore the first sum provides the independent domination polynomial with some additional constant terms. Hence, it is enough to subtract the constant factor in the first sum. $\square$

Lemma 4.15. *Let $G = (V, E)$ be a graph. Then*

$$\min_{W \in \mathrm{Ess}_i(G)} \{|W|\} = \delta(G)$$

and

$$N(v) \in \mathrm{Ess}_i(G), v \in V.$$

Proof. Suppose there exists a set U in $\mathrm{Ess}_i(G)$ with $|U| < \delta(G)$. Then a vertex v with $N(v) \subseteq U$ in G exists. But if such a vertex exists, then it has a degree less than $\delta(G)$, which is a contradiction.

The second equation follows from the fact that for every vertex v in G the open neighborhood of this vertex is the smallest subset which fulfills the essential set condition. $\square$

The following lemma finds usage in the calculation of the independent domination polynomial.

Lemma 4.16. *Let $G = (V, E)$ be a graph and $W, U \subseteq V$ be vertex subsets of G. If W is an independent dominating set of the graph G and $W \subset U$, then U is not an independent dominating set.*

Proof. Let $W \subseteq V$ be an independent dominating set of the graph, then every vertex in $V \backslash W$ is adjacent to a vertex in W. Therefore, every vertex subset $W \cup \{v\}$, for $v \in V \backslash W$, has at least one adjacent pair of vertices and hence it is not independent in G. □

Corollary 4.17. *Let $G = (V, E)$ be a graph and S be the partial ordered set $(\mathcal{P}(V), \subseteq)$. Then the set of the independent dominating sets of G is an anti-chain in S.*

Theorem 4.18. *Let $G = (V_1 \cup V_2 \cup \{v\}, E)$ be a graph and $v \notin V_1 \cup V_2$ be an articulation of G. Furthermore, let $V_1 \cap V_2 = \emptyset$, $W_1 = N(v) \cap V_1 \neq \emptyset$ and $W_2 = N(v) \cap V_2 \neq \emptyset$, such that there is no edge between a vertex of V_1 and a vertex of V_2 in G. Then*

$$\begin{aligned} \mathrm{D}_i(G, x) =& x\, \mathrm{D}_i(G_1 - N[v], x)\, \mathrm{D}_i(G_2 - N[v], x) \\ &+ \sum_{\substack{W \subseteq W_1 \cup W_2 \\ W \neq \emptyset \\ \mathrm{iso}(G[W]) = |W|}} (-1)^{|W|+1} x^{|W|}\, \mathrm{D}_i(G_1 - N[W], x)\, \mathrm{D}_i(G_2 - N[W], x). \end{aligned}$$

Proof. If the articulation v is dominating, then all adjacent vertices are dominated and they cannot be dominating. Therefore, this case will be counted by $x\, \mathrm{D}_i(G - N[v], x)$. If the vertex v is not dominating, then at least one neighbor must be dominating. But these vertices must also form an independent set in G. With the principle of inclusion-exclusion and Lemma 4.2 the theorem follows. □

Corollary 4.19. *Let $G = (V_1 \cup V_2 \cup \{v\}, E)$ be a graph, v an articulation of G and $V_2 \subset N(v)$. Then*

$$\mathrm{D}_i(G, x) = x\, \mathrm{D}_i(G_1 - N[v], x) + \mathrm{D}_i(G_1, x)\, \mathrm{D}_i(G_2, x).$$

Proof. The first part of the sum follows directly from the first part of the sum in Theorem 4.18. If the articulation is non-dominating, then at least one vertex in G_2 must be dominating. Hence, the articulation will be dominated from the dominating vertices in G_2 and the theorem follows. □

4.1 Recurrence Equations

Definition 4.20. *Let $G = (V, E)$ be a graph and $u \in V$. Then $p_u^i(G)$ is the independent domination polynomial of $G - N[u]$ under the condition that all vertices in $N(u)$ are dominated from a vertex in $G - N[u]$.*

Remark 4.21. *Let $G = (V, E)$ be a graph and $v \in V$. Then*

$$p_v^i(G, x) = p_v^i(G \odot v, x). \tag{4.3}$$

Theorem 4.22. *Let $G = (V, E)$ be a graph and v be a vertex of the graph. Then*

$$\mathrm{D}_i(G, x) = \mathrm{D}_i(G - v, x) - p_v^i(G) + x\, \mathrm{D}_i(G - N[v], x).$$

Proof. If the vertex v is dominating, then it dominates all vertices in the neighborhood and these vertices cannot be dominating. This case will be counted by $x \operatorname{D}_i(G - N[v], x)$. If the vertex v is not dominating, then at least one vertex in $N(v)$ must be dominating. The polynomial $\operatorname{D}_i(G - v, x)$ counts these independent dominating sets, but it also counts those sets where no vertex is dominating in $N(v)$. Therefore, we must subtract the polynomial for these cases to obtain the theorem. $\qquad\square$

Corollary 4.23. *Let $G = (V, E)$ be a graph, $u, v \in V$ and $N(u) \subseteq N(v)$. Then*

$$\operatorname{D}_i(G, x) = \operatorname{D}_i(G - v, x) - x p_v^i(G - N[u]) + x^2 \operatorname{D}_i(G - N[v] - u, x).$$

Corollary 4.24. *Let $G = (V, E)$ be a graph, $u, v \in V$ and $N(u) = N(v)$. Then*

$$\operatorname{D}_i(G, x) = \operatorname{D}_i(G - v, x) + (x^2 - x) \operatorname{D}_i(G - N[v] - u, x).$$

Theorem 4.25. *Let $G = (V, E)$ be a graph and $v \in V$. Then*

$$\operatorname{D}_i(G, x) = \operatorname{D}_i(G - v, x) + \operatorname{D}_i(G \odot v, x) - \operatorname{D}_i(G \circledcirc v, x).$$

Proof. To prove the theorem we use the idea of the proof of Theorem 5.22. Applying Equations (4.3) to Theorem 4.22 yields

$$\begin{aligned}
\operatorname{D}_i(G, x) - \operatorname{D}_i(G - v, x) &= x \operatorname{D}_i(G - N[v], x) - p_v^i(G) \\
&= x \operatorname{D}_i(G - N[v], x) - p_v^i(G \odot v).
\end{aligned} \tag{4.4}$$

Now applying Theorem 4.22 to the graph $G \odot v$ leads to

$$\operatorname{D}_i(G \odot v, x) - \operatorname{D}_i((G \odot v) - v, x) = x \operatorname{D}_i((G \odot v) - N[v], x) - p_v^i(G \odot v). \tag{4.5}$$

The Equations (4.4) and (4.5) together yield the theorem. $\qquad\square$

The graph $G \circ v$ can be obtained from the graph G by removing v and adding a loop to every neighbor of v. A loop in the context of domination means that the vertex dominates itself. If a vertex v has a loop, then $v \in N(v)$. Therefore $\operatorname{D}_i(G \circ v, x)$ is the independent domination polynomial of the graph $G - v$ under the condition that no vertex in $N(v)$ is dominating. Together with Theorem 4.22 we obtain the following corollary.

Corollary 4.26. *Let $G = (V, E)$ be a graph (loops allowed) and v be a vertex of the graph. Then*

$$\operatorname{D}_i(G, x) = \begin{cases} x \operatorname{D}_i(G - N[v], x) + \operatorname{D}_i(G - v, x) - \operatorname{D}_i(G \circ v, x), & \text{if } v \notin N(v) \\ \operatorname{D}_i(G - v, x) - \operatorname{D}_i(G \circ v, x), & \text{otherwise.} \end{cases}$$

It is also possible to prove a theorem which yields a recurrence equation for the deletion of an edge in the graph.

Theorem 4.27. *Let $G = (V, E)$ be a graph and $e = \{u, v\}$ be an edge of the graph. Then*

$$\operatorname{D}_i(G, x) = \operatorname{D}_i(G - e, x) - x^2 \operatorname{D}_i(G - N[u, v], x) + x \operatorname{D}_i(G \circ v - N[u], x) + x \operatorname{D}_i(G \circ u - N[v], x).$$

Proof. Every independent dominating set from G is an independent dominating set in $G - e$, except for some sets where either u or v are dominating. On the other hand, both vertices u and v can be dominating in $G - e$, but not in G. Hence, we must subtract $x^2 \, \mathrm{D}_i(G - N[u,v], x)$. Suppose now that only one of these two vertices is dominating and no vertex in the neighborhood of the other vertex is dominating. This situation will be counted in the graph G but not in the graph $G - e$. Therefore, we must add the polynomial for this case and the theorem follows. Note that $x \, \mathrm{D}_i(G \circ v - N[u], x)$ is the independent domination polynomial under the condition that the vertex u is dominating and no vertex in the neighborhood of v (except for u) is dominating. $\square$

Corollary 4.28. *Let $G = (V, E)$ be a graph, $e = \{u, v\}$ be an edge of the graph and $N[u] = N[v]$. Then*

$$\mathrm{D}_i(G, x) = \mathrm{D}_i(G - e, x) + (2x - x^2) \, \mathrm{D}_i(G - N[u], x).$$

4.2 Non-Isomorphic Graphs

An interesting question is: How well does the independent domination polynomial distinguishes non-isomorphic graphs? This is of interest especially in comparison to the domination polynomial and the independence polynomial. Figure 4.1 shows the smallest pair of non-isomorphic connected graphs G_1 and G_2 with the same independent domination polynomial. These two graphs also have the same independence polynomial, but different domination polynomials:

$$\mathrm{D}_i(G_1, x) = \mathrm{D}_i(G_2, x) = x^3 + 2x^2$$
$$\mathrm{I}(G_1, x) = \mathrm{I}(G_2, x) = x^3 + 5x^2 + 5x + 1$$
$$\mathrm{D}(G_1, x) = x^5 + 5x^4 + 8x^3 + 3x^2$$
$$\mathrm{D}(G_2, x) = x^5 + 5x^4 + 9x^3 + 4x^2$$
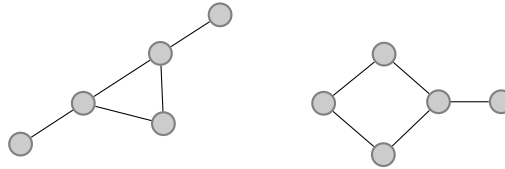
Fig. 4.1: The smallest pair of non-isomorphic graphs with the same independent domination polynomial.

In the case of trees, Figure 4.2 shows two non-isomorphic trees with ten vertices with the same independent domination polynomial. It can be shown by computer research that this is the smallest non-isomorphic pair. They also have the same independence polynomial, but not the same domination polynomial. In [DPT03] Dohmen, Pönitz and Tittmann gave a pair of non-isomorphic trees having the same independence polynomial but different independent domination polynomials and different domination polynomials.
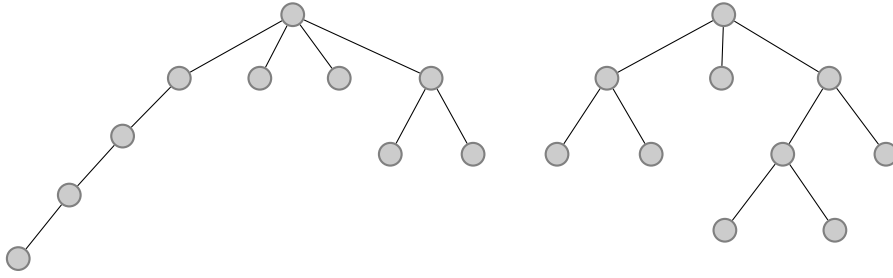
Fig. 4.2: The smallest pair of non-isomorphic trees with the same independent domination polynomial.

4.3 Graph Products

4.3.1 Cartesian Product

Theorem 4.29. *Let H be a simple graph with n vertices and K_m be a complete graph with $m \geq \Delta(H) + 1$ vertices. Then*

$$D_i(H \Box K_m, x) = \chi(H, m)x^n.$$

Proof. It is enough to show that every proper coloring with m colors of the graph H represents an independent dominating set in $H \Box K_m$ and vice versa. Let $C : V(H) \to \{1, \ldots, m\}$ be such a proper coloring and let $c(v)$ be the color of the vertex $v \in H$. Then the vertex subset $\{(v, c(v)) : \forall v \in V(H)\}$ is an independent dominating set of $H \Box K_m$. Every independent dominating set can also be represented by a proper coloring because of the fact that in every row exactly one vertex is dominating and no adjacent vertices are dominating. The number of proper colorings with m colors is given by the evaluation of the chromatic polynomial and therefore the theorem follows. $\qquad\square$

The following three corollaries are a direct consequence of this theorem.

Corollary 4.30. *Let K_n and K_m be two complete graphs and let $n \leq m$. Then*

$$D_i(K_n \Box K_m, x) = m^{\underline{n}} x^n.$$

Remark 4.31. $D_i(K_n \Box K_m, 1)$ *is the number of dominating non-attacking rooks on a $n \times m$-chessboard.*

Corollary 4.32. *Let K_n be a complete graph with n vertices, $n \geq 3$, and P_m be a path with m vertices. Then*

$$D_i(K_n \Box P_m, x) = n(n-1)^{m-1} x^m.$$

Corollary 4.33. *Let K_n be a complete graph with $n \geq 3$ vertices and C_m be a cycle with $m \geq 2$ vertices. Then*

$$D_i(K_n \Box C_m, x) = ((n-1)^m + (-1)^m(n-1))x^m.$$

The independent domination polynomials of the product $K_n \Box P_m$ and $K_n \Box C_m$ have nice combinatorial interpretations. The independent domination polynomial of the first product yields the number of words of length m with n letters and no two adjacent identical letters. In case of the product $K_n \Box C_m$, also the first and the last letter are different. For $n = 4$, this sequence can be found as A218034 in the OEIS [OEI14].

Theorem 4.34. *Let H be a simple graph with m vertices and K_n be a complete graph with $n \leq \Delta(H)$ vertices. Assume H has an unique vertex v with $N[v] = V(H)$ and $\deg(w) \leq n-1$, for all $w \in V(H)\backslash\{v\}$. Then*

$$\mathrm{D}_i(K_n \square H, x) = \chi(H, n)x^m + x^{m-1}\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}\chi(H-v, n-k).$$

Proof. The independent dominating sets of size m are counted by $\chi(H, n)$. But it is also possible to find independent dominating sets of size $m-1$ which do not include the vertices (x, v). More precisely, the vertices (x, v), for all $x \in \{1, \ldots, n\}$, are dominated if at least one vertex in every set $\{(x, u) : u \in N_H(v)\}$, for all $x \in \{1, \ldots, n\}$, is dominating. The evaluation of the chromatic polynomial $\chi(H-v, n)$ yields the number of ways to color the vertices in $V(H)\backslash\{v\}$ with n colors. But now we count also those colorings where no vertex is chosen from $\{(x, u) : u \in N_H(v)\}$ for some $x \in \{1, \ldots, n\}$. With the principle of inclusion-exclusion, the theorem follows. $\qquad\square$

The last two theorems have some restrictions to the maximum degree in the graph H. Therefore, it would be useful to have a more general version of these theorems. The next two theorems yield the independent domination polynomial of the product graphs $K_2 \square H$ and $K_3 \square H$, respectively, for which H is an arbitrary graph.

Theorem 4.35. *Let K_2 be a complete graph with two vertices and $H = (V, E)$ be a graph with n, $n \geq 2$, vertices. Then*

$$\mathrm{D}_i(K_2 \square H, x) = \sum_{\substack{W \subseteq V \\ W \neq \emptyset \\ W \text{ ind.} \\ V\backslash N_H[W] \text{ ind.}}} x^{n-|N_H(W)|}\,\mathrm{D}_i(H - W - N_H[V - N_H[W]], x).$$

Proof. Let W and $U = V\backslash N_H[W]$ be independent vertex subsets of H. Then the vertices in the sets $\{(1, w) : w \in W\}$ and $\{(2, u) : u \in U\}$ dominate all vertices in the product graph except for the vertices in $\{(2, v) : v \in N_H(W)\backslash N_H(U)\}$. These are exactly those vertices in the second row, which are not adjacent to the vertices in $\{(2, u) : u \in U\}$. Therefore, any independent dominating set of these vertices together with the vertices in W and U form an independent dominating set in $K_2 \square H$. $\qquad\square$

It is possible to generalize the idea of the last theorem to products of a K_3 with an arbitrary graph.

Theorem 4.36. *Let K_3 be a complete graph with three vertices, H be a non-empty graph with n, $n \geq 3$, vertices and $N[W] = N_H[W]$. Then*

$$\mathrm{D}_i(K_3 \square H, x) = \sum_{\substack{W \subseteq V \\ W \neq \emptyset \\ W \text{ ind.}}} x^{n-|N(W)|} \sum_{\substack{U \subseteq V\backslash N[W] \\ U \text{ ind.} \\ (V\backslash N[W])\backslash U \text{ ind.}}} \sum_{\substack{Z \subseteq N(W)\backslash N(U) \\ Z \cup U \text{ ind.} \\ (Y\backslash N[Z])\cup X \text{ ind.}}} x^{|Z|+|Y\backslash N[Z]|} \times$$
$$\times\, \mathrm{D}_i(H[N(W) - Z - N[X] - N[Y - N[Z]], x),$$

with $X = (V\backslash N[W])\backslash U$ and $Y = N(W)\backslash N(U)$.

Proof. The idea of the theorem is to choose a non-empty independent set W in the first row of the product graph. The vertices in W are the only dominating vertices in the first row (see Figure 4.3). The non-dominated vertices in the first row are dominated by the vertices in the sets U and X in the second resp. third row. From the subset $Y = N(W)\backslash N(U)$ of the non-dominated vertices in the second row, we choose again an independent subset Z, which has the property that the vertex subset $Y\backslash N[Z]$ is also an independent set. The only vertices which now are non-dominated are the vertices in the third row, which are in $N(W)$ and adjacent to Z, but not adjacent to $Y\backslash N[Z]$. For the induced subgraph of these vertices, we calculate the independent domination polynomial and the theorem follows. $\square$
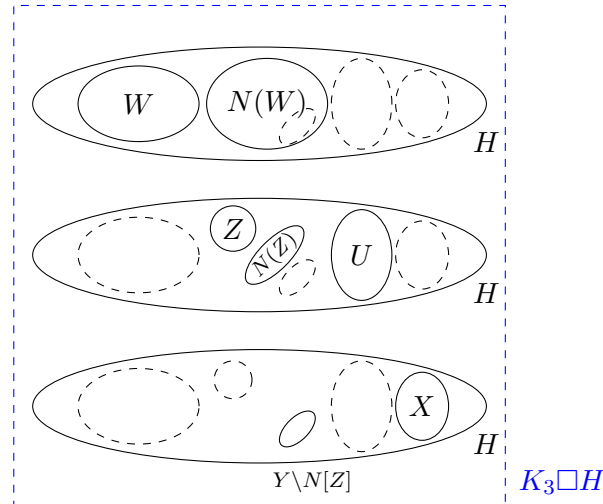
Fig. 4.3: Illustration of the proof of Theorem 4.36.

In the case of $P_m\square P_n$, not much is known about independent dominating sets, but it is already known (see [Weic]) that the independence number of such products is simply

$$\alpha(P_m\square P_n) = \left\lceil \frac{mn}{2} \right\rceil.$$

The calculation of the independent domination number is much more complicated. There are only few easy cases known. If $m = 3$, then we have (see [Cor91])

$$\gamma_i(P_3\square P_n) = 2\left\lfloor \frac{n-2}{2} \right\rfloor + 2.$$

To prove a recurrence equation for the independent domination polynomial of the product graph $P_3\square P_n$, we distinguish which vertices in the last column are dominating. The first observation is that at least one vertex in the last column must be dominating. If this is not the case these three vertices cannot be dominated by an independent dominating set. If the vertex $(1, n)$ is dominating, then the vertices $(1, n-1)$ and $(2, n)$ are dominated. The remaining graph will be denoted by G_n^1 (see Figure 4.4(a)). If the vertex $(2, n)$ is dominating, then the vertices $(1, n)$, $(3, n)$ and $(2, n-1)$ are dominated. The remaining graph will be denoted by G_{n-1}^2 (see Figure 4.4(b)). If the vertices $(1, n)$ and $(3, n)$ are dominating, then the vertices $(1, n-1)$, $(2, n)$ and $(3, n-1)$ are dominated and this graph will be denoted by G_{n-1}^3 (see Figure 4.4(c)).
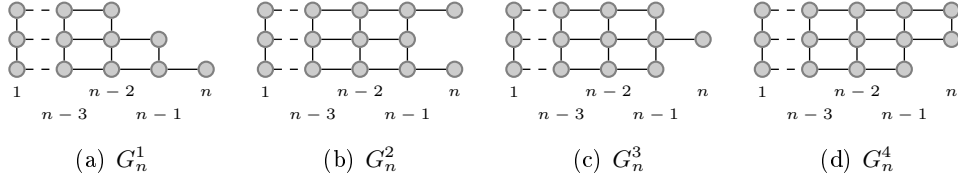
Fig. 4.4: Graphs G_n^1, G_n^2, G_n^3 and G_n^4

This case distinction leads directly to a recurrence equation for the independent domination polynomial:

$$\mathrm{D}_i(P_3 \square P_n, x) = 2x\,\mathrm{D}_i(G_n^1, x) + x\,\mathrm{D}_i(G_{n-1}^2, x) + x^2\,\mathrm{D}_i(G_{n-1}^3, x). \tag{4.6}$$

The following theorem provides recurrence equations for the graphs G_n^1, G_n^2 and G_n^3.

Lemma 4.37. *Let G_n^1, G_n^2 and G_n^3 be the graphs defined above and the graph G_n^4 be obtained from the graph $P_3 \square P_n$ by deleting the vertex $(3,n)$ (see Figure 4.4(d)). Then*

$$
\begin{aligned}
\mathrm{D}_i(G_n^1, x) &= x\left(\mathrm{D}_i(G_{n-1}^3, x) + \mathrm{D}_i(G_{n-2}^4, x)\right), \\
\mathrm{D}_i(G_n^2, x) &= x^2\left(\mathrm{D}_i(G_{n-1}^3, x) + 2\,\mathrm{D}_i(G_{n-2}^4, x) + \mathrm{D}_i(G_{n-2}^3, x)\right), \\
\mathrm{D}_i(G_n^3, x) &= x\left(\mathrm{D}_i(G_{n-1}^2, x) + \mathrm{D}_i(G_{n-2}^2, x)\right), \\
\mathrm{D}_i(G_n^4, x) &= x\left(\mathrm{D}_i(G_{n-1}^4, x) + \mathrm{D}_i(G_{n-1}^2, x)\right).
\end{aligned}
$$

The initial conditions are

$$
\begin{aligned}
\mathrm{D}_i(G_3^1, x) &= x^4 + 3x^2, \\
\mathrm{D}_i(G_1^2, x) &= x^2, & \mathrm{D}_i(G_2^2, x) &= x^3 + 3x^2, \\
\mathrm{D}_i(G_1^3, x) &= x, & \mathrm{D}_i(G_2^3, x) &= x^3 + x, \\
\mathrm{D}_i(G_1^4, x) &= 2x.
\end{aligned}
$$

Proof. We use again case distinction for the four types of graphs.

G_n^1: If the vertex $(3,n)$ is dominating, then the vertex $(3, n-1)$ is dominated. This leads to $x\,\mathrm{D}_i(G_{n-1}^3, x)$. If the vertex $(3,n)$ is non-dominating, then the vertex $(3, n-1)$ must be dominating. This case will be counted with $x\,\mathrm{D}_i(G_{n-2}^4, x)$.

G_n^2: Notice that exactly one of the vertices $(1,n)$ and $(1, n-1)$ and exactly one of the vertices $(3,n)$ and $(3, n-1)$ must be dominating. There are four possible choices of two vertices which fulfill these conditions. These different choices lead to the recurrence equation for this special graph.

G_n^3: If the vertex $(2,n)$ is dominating, then the vertex $(2, n-1)$ is dominated and the remaining graph is the graph G_{n-1}^2. Therefore, this case will be counted by $x\,\mathrm{D}_i(G_{n-1}^2, x)$. If the vertex $(2,n)$ is non-dominating, then the vertex $(2, n-1)$ must be dominating. In this case the vertices $(1, n-1)$, $(2, n-2)$, $(2,n)$ and $(3, n-1)$ are dominated and this will be counted by $x\,\mathrm{D}_i(G_{n-2}^2, x)$.

G_n^4: Notice that exactly one of the two vertices $(1,n)$ and $(2,n)$ must be dominating. If the vertex $(1,n)$ is dominating, then the vertices $(1, n-1)$ and $(2,n)$ will be dominated and this will be counted by $x\,\mathrm{D}_i(G_{n-1}^4, x)$. If the vertex $(2,n)$ is dominating, then the vertices $(1,n)$ and $(2, n-1)$ are dominated. This leads to the graph G_{n-1}^2 and will be counted by $x\,\mathrm{D}_i(G_{n-1}^2, x)$. $\qquad\square$

In the Equation (4.6) the independent domination polynomial of the graph G_n^1 can be replaced with the equation in the previous lemma. This provides a recurrence equation which only depends on the graphs G_n^2, G_n^3 and G_n^4:

$$\mathrm{D}_i(P_3 \square P_n, x) = x^2 \mathrm{D}_i(G_{n-1}^3, x) + x \mathrm{D}_i(G_{n-1}^2, x) + 2x^2 \mathrm{D}_i(G_{n-2}^4, x).$$

The problem with this method is that for larger m the recurrence equations get more complicated and it is therefore not practicable for such cases.

4.3.2 Tensor Product

The first theorem provides an equation for the independent domination polynomial of the tensor product of two complete graphs.

Theorem 4.38. *Let K_n and K_m be two complete graphs. Then*

$$\mathrm{D}_i(K_m \times K_n, x) = mx^n + nx^m.$$

Proof. If a vertex of the graph is dominating, then all vertices are dominated except for the vertices in the same row and column of the chosen vertex. Let (v, w) be this dominating vertex. It is only possible to choose an additional dominating vertex in the row R_v or the column w. If we choose a vertex in the row R_v, then all vertices in the column w are dominated. Therefore, all non-dominated vertices in the row R_v must be dominating. So we have m ways to choose a column with n vertices and n ways to choose a row with m vertices. $\square$

The second result is about the product of a path with a complete graph. Let G_n^m be the graph obtained from the product graph $P_{n-1} \times K_m$ by adding an additional vertex which is adjacent to all but one vertices in the first row (see Figure 4.5).

Theorem 4.39. *Let P_n be a path with n vertices, $n \geq 4$, and K_m be a complete graph with m vertices ($m \geq 2$). Then*

$$\mathrm{D}_i(P_n \times K_m, x) = x^m \mathrm{D}_i(P_{n-2} \times K_m, x) + x^m \mathrm{D}_i(P_{n-3} \times K_m, x) + mx^2 \mathrm{D}_i(G_{n-2}^m, x),$$

with the initial conditions

$$\mathrm{D}_i(P_1 \times K_m, x) = x^m \text{ and } \mathrm{D}_i(P_2 \times K_m, x) = mx^2 + 2x^m.$$

Proof. If no vertex is dominating in the first row, then in the second row at least two vertices must be dominating and therefore all vertices in the third row are dominated. The non-dominated vertices in the second row can only be dominated by themselves. This leads to the term $x^m \mathrm{D}_i(P_{n-3} \times K_m, x)$. If all vertices are dominating in the first row, then all vertices in the second row are dominated. This yields the term $x^m \mathrm{D}_i(P_{n-2} \times K_m, x)$. If in the first row exactly one vertex is dominating, then in the second row all but one vertices are dominated, but this one non-dominated vertex must be dominating. Otherwise, the remaining vertices in the first row cannot be dominated. The single dominating vertex in the second row dominates all but one vertices in the third row. The remaining graph is the graph G_{n-2}^m. $\square$

A recurrence equation for the independent domination polynomial of the graph G_n^m is proved in the following.
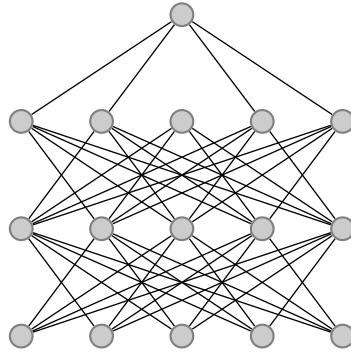
Fig. 4.5: The graph G_4^5.

Lemma 4.40. *Let G_n^m be the graph just defined, $n \geq 4$ and $m \geq 2$. Then*

$$\mathrm{D}_i(G_n^m, x) = x\,\mathrm{D}_i(G_{n-1}^m, x) + (m-1)x^2\,\mathrm{D}_i(G_{n-3}^m, x) + x^m\,\mathrm{D}_i(P_{n-3} \times K_m, x),$$

with the initial conditions

$$\mathrm{D}_i(G_1^m, x) = x \text{ and } \mathrm{D}_i(G_2^m, x) = x^2 + x^m.$$

Proof. The proof uses the same idea as the proof of Theorem 4.39.   □

We can use the same idea to prove a result for the product $C_n \times K_m$, which is slightly more complex. For this equation, we need some special graphs again. Let H_n^m be the graph constructed from the graph $P_{n-2} \times K_m$ by adding two additional vertices u and v, which are adjacent to all but one vertices in the first and the last row, respectively. The two non-adjacent vertices are in the same column of the product graph (see Figure 4.6). The graph I_n^m is constructed in the same way except for the fact that the two non-adjacent vertices of u and v in the first and the last row are not in the same column.
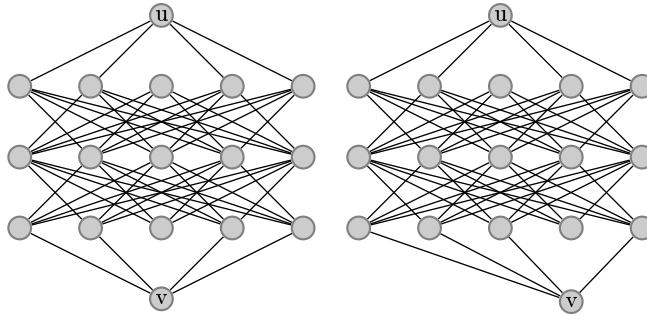
Fig. 4.6: The graphs H_5^5 (left) and I_5^5 (right).

Theorem 4.41. *Let C_n be a cycle with n vertices ($n \geq 7$) and K_m be a complete graph with m vertices ($m \geq 2$). Then*

$$\begin{aligned}
\mathrm{D}_i(C_n \times K_m, x) =&\, 2x^m\,\mathrm{D}_i(P_{n-3} \times K_m, x) + x^{2m}\,\mathrm{D}_i(P_{n-6} \times K_m, x) \\
&+ mx^2\left[2(m-1)x^2\,\mathrm{D}_i(I_{n-5}^m, x) + \mathrm{D}_i(H_{n-2}^m, x) + 2x^m\,\mathrm{D}_i(G_{n-5}^m, x)\right].
\end{aligned}$$

Proof. The proof uses the same idea as the proof of Theorem 4.39.   □

Lemma 4.42. *Let H_n^m and I_n^m be the graphs just defined. Furthermore, let $n \geq 4$ and $m \geq 2$. Then*

$$\mathrm{D}_i(H_n^m, x) = x\, \mathrm{D}_i(H_{n-1}^m, x) + x^m\, \mathrm{D}_i(G_{n-3}^m, x) + (m-1)x^2\, \mathrm{D}_i(I_{n-3}^m, x),$$
$$\mathrm{D}_i(I_n^m, x) = x\, \mathrm{D}_i(I_{n-1}^m, x) + x^m\, \mathrm{D}_i(G_{n-3}^m, x)$$
$$+ (m-2)x^2\, \mathrm{D}_i(I_{n-3}^m, x) + x^2\, \mathrm{D}_i(H_{n-3}^m, x),$$

with the initial conditions

$$\mathrm{D}_i(H_1^m, x) = 0, \quad \mathrm{D}_i(H_2^m, x) = x^2,$$
$$\mathrm{D}_i(I_1^m, x) = 1 \text{ and } \mathrm{D}_i(I_2^m, x) = 2x.$$

Proof. The proof uses the same idea as the proof of Theorem 4.39. □

The tensor product of two connected graphs is sometimes non-connected. Weichsel proved the following theorem in 1962, which yields a characterization for such non-connected product graphs.

Theorem 4.43. *[Wei62] Let G and H be connected graphs. The graph $G \times H$ is connected if and only if either G or H contains an odd cycle. The graph $G \times H$ has exactly two components if G and H are bipartite.*

The following lemma is a consequence of this theorem.

Lemma 4.44. *[Klo99a] The tensor product $P_n \times P_k$, $n, k \geq 2$, consists of two components. Moreover, if $k = 2$, the components consist of two paths of length n. If both k and n are odd, these components are not isomorphic. If at least one of these two numbers is even, the components are isomorphic.*

A direct consequence of the previous lemma is that the independent domination polynomial of P_n and P_2 can be calculated by

$$\mathrm{D}_i(P_n \times P_2, x) = \mathrm{D}_i(P_n, x)^2.$$

For the product $P_n \times P_3$ we distinguish two cases. If n is odd, then the first component, denoted by G_n'', consists of $\lfloor \frac{n}{2} \rfloor$ connected cycles (see the blue vertices in Figure 4.7(a)) and the second component, denoted by G_n''', consists of $\lfloor \frac{n}{3} \rfloor$ connected cycles and four additional hanging vertices (see the red vertices in Figure 4.7(a)). If n is even, then the two components are isomorphic and consist of $\lfloor \frac{n}{3} \rfloor$ connected cycles and two additional hanging vertices (see Figure 4.7(b)). Such a component will be denoted by G_n'.

First we prove recurrence equations for these three different graphs.

Lemma 4.45. *Let G_n', G_n'' and G_n''' be the graphs just defined, with $n \geq 4$ and $m \geq 2$. Then*

$$\mathrm{D}_i(G_n', x) = x\, \mathrm{D}_i(G_{n-2}', x) + x^2\, \mathrm{D}_i(G_{n-3}''', x),$$
$$\mathrm{D}_i(G_n'', x) = x\, \mathrm{D}_i(G_{n-2}'', x) + x^2\, \mathrm{D}_i(G_{n-3}', x),$$
$$\mathrm{D}_i(G_n''', x) = x^2\, \mathrm{D}_i(G_{n-2}''', x) + x\, \mathrm{D}_i(G_{n-3}', x).$$

The initial conditions are

$$\mathrm{D}_i(G_2', x) = \mathrm{D}_i(P_3, x) = x^2 + x,$$
$$\mathrm{D}_i(G_3'', x) = \mathrm{D}_i(C_4, x) = 2x^2,$$
$$\mathrm{D}_i(G_1''', x) = x^2,$$
$$\mathrm{D}_i(G_3''', x) = \mathrm{D}_i(S_5, x) = x^5 + x.$$

(a) Tensor product $P_5 \times P_3$ (b) Tensor product $P_6 \times P_3$
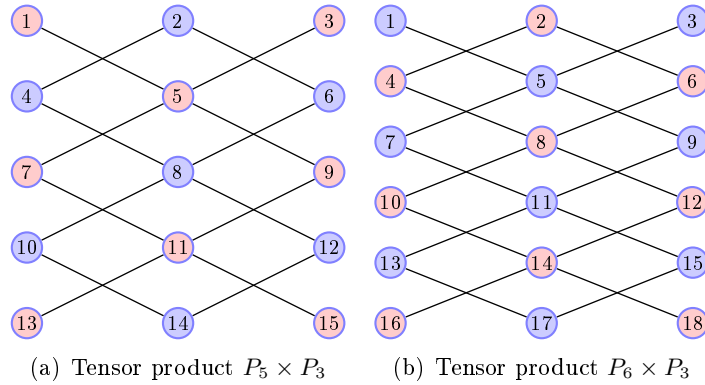
Fig. 4.7: Tensor products of two paths.

Proof. The proof uses the same idea as the proof of Theorem 4.39.   □

Lemma 4.45 together with the previous considerations yields the following theorem.

Theorem 4.46. *Let P_n be a path with n vertices ($n \geq 2$). Then*

$$\mathrm{D}_i(P_n \times P_3, x) = \begin{cases} \mathrm{D}_i(G'_n, x)^2, & \text{if n is even} \\ \mathrm{D}_i(G''_n, x)\,\mathrm{D}_i(G'''_n, x), & \text{otherwise.} \end{cases}$$

4.3.3 Lexicographic Product

Theorem 4.47. *Let G and H be graphs. Then*

$$\mathrm{D}_i(G \cdot H, x) = \mathrm{D}_i(G, \mathrm{D}_i(H, x)).$$

Proof. Let $W \subseteq V(G)$ be an independent dominating set in G. Then the set $W' = \{(w, 1) : w \in W\}$ dominates all vertices in $\{(v, u) : v \in V(G)\backslash W, u \in V(H)\}$ and $\{(w, v) : w \in W, v \in N_H[1]\}$. Let $I \subseteq V(H)$ be an independent dominating set in H. Now we construct the set W'' from W' by replacing the vertex $(v, 1) \in W'$ with the vertices (v, w), for $w \in I$. This set W'' is an independent dominating set in the product graph and every independent dominating set can be constructed in this way. Therefore, the theorem follows.   □

A direct consequence of the last theorem is the following corollary. This result can also be found in [NR96].

Corollary 4.48. *Let G and H be graphs. Then*

$$\gamma_i(G \cdot H, x) = \gamma_i(G)\gamma_i(H) \text{ and}$$
$$\alpha(G \cdot H, x) = \alpha(G)\alpha(H).$$

Corollary 4.49. *Let $G = (V, E)$ be a graph and K_n be the complete graph with n vertices. Then*

$$\mathrm{D}_i(G \cdot K_n, x) = \mathrm{D}_i(G, nx).$$

Corollary 4.50. *Let $G = (V, E)$ be a graph and K_n be the complete graph with n vertices. Then*

$$\mathrm{D}_i(K_n \cdot G, x) = n\,\mathrm{D}_i(G, x).$$

For the product of P_n and C_n with an arbitrary graph G, it is also possible to prove recurrence equations with respect to smaller product graphs.

Remark 4.51. *The independent domination polynomial* $D_i(P_n \cdot G, x)$ *equals one for* $n = 0$.

Theorem 4.52. *Let* $G = (V, E)$ *be a graph and* P_n *the path with* $n \geq 3$ *vertices. Then*

$$D_i(P_n \cdot G, x) = D_i(G, x) \left(D_i(P_{n-2} \cdot G, x) + D_i(P_{n-3} \cdot G, x) \right).$$

Proof. If at least one vertex of the first row is in the dominating set, then it dominates all vertices in the next row and there cannot be a dominating vertex in this row. If the dominating vertices of the first row only form a partial dominating set with respect to the vertices of this row (one copy of G), then the non-dominated vertices in the first row cannot be dominated because all neighbors of these vertices are adjacent to an already dominating vertex. This case will be counted by $D_i(G, x) D_i(P_{n-2} \cdot G, x)$. If there is no dominating vertex in the first row, then these vertices can only be dominated by a vertex of the second row. Therefore, the dominating vertices in the second row must be an independent dominating set. This will be counted by $D_i(G, x) D_i(P_{n-3} \cdot G, x)$ and the theorem follows.   □

Theorem 4.53. *Let* $G = (V, E)$ *be a graph and* C_n *be the cycle with* n *vertices* ($n \geq 6$). *Then*

$$D_i(C_n \cdot G, x) = D_i(G, x) \left(2 D_i(P_{n-3} \cdot G, x) + D_i(G, x) D_i(P_{n-6} \cdot G, x) \right).$$

Proof. If the vertices in the first row form an independent dominating set, then all vertices in the second and in the last row are dominated. This case is counted by $D_i(G, x) D_i(P_{n-3} \cdot G, x)$. If no vertex is dominating in the first row, then at least one vertex in the second or the last row must be dominating. If the vertices in the second row form an independent dominating set, then all vertices in the first and in the third row are dominated. This case is again counted by $D_i(G, x) D_i(P_{n-3} \cdot G, x)$. The last case is that in the first and in the second row no vertex is dominating. Then at least one vertex in the last row and at least one vertex in the third row must be dominating. This yields $D_i(G, x) D_i(P_{n-6} \cdot G, x)$ and the theorem follows.   □

4.3.4 Strong Product

Klobučar [Klo05] showed that the independent domination number of the strong product of two paths $\gamma_i(P_m \boxtimes P_n)$ is equal to $\lceil \frac{m}{3} \rceil \lceil \frac{n}{3} \rceil$ and the independence number $\alpha(P_m \boxtimes P_n)$ is equal to $\lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$. If m and n are odd, then the number of maximal independent dominating sets is equal to one. This follows directly from the results in [Klo05].

Theorem 4.54. *Let* $G = (V, E)$ *be a graph and* K_n *be the complete graph with* n *vertices. Then*

$$D_i(K_n \boxtimes G, x) = D_i(G, nx).$$

Proof. The proof is analog to the proof of Theorem 4.47 and the Corollary 4.49.   □

4.4 Special Graph Classes

As mentioned before, the calculation of the independent domination polynomial can be done easily in some special graph classes. The independent domination polynomial of the edgeless graph E_n is simply x^n and for the complete graph, it is given by

$$D_i(K_n, x) = nx. \tag{4.7}$$

Theorem 4.55. *Let $G = (V, E)$ be a graph obtained from a complete graph K_n by removing all edges of a matching M of size k. Then*

$$\mathrm{D}_i(G, x) = (n - 2k)x + kx^2.$$

Proof. There are $n - 2k$ vertices in the graph G of degree $n - 1$ and therefore each of these vertices dominates the other vertices of the graph. This is counted by $(n - 2k)x$. If a vertex incident to a matching edge is dominating, then this vertex dominates all vertices of the graph except for the other vertex of the matching edge. Therefore, this vertex has to be dominating and we have k possibilities to choose such a pair. □

Theorem 4.56. *Let $K_{pq} = (V_1 \cup V_2, E)$ be a complete bipartite graph. Then*

$$\mathrm{D}_i(K_{pq}, x) = x^p + x^q.$$

Proof. If at least one vertex is dominating in V_1, then all vertices are dominated in V_2. Hence, all vertices in V_1 must be dominating so that they are a dominating set in the graph. The same argumentation holds if at least one vertex in V_2 is dominating. □

Corollary 4.57. *Let $G = (V, E)$ be a graph obtained from a complete bipartite graph K_{pq} by removing all edges of a matching M of size k. Then*

$$\mathrm{D}_i(G, x) = x^p + x^q + kx^2.$$

Simple k-paths form another interesting graph class. If a vertex in a simple k-path is dominating, then it dominates the next k and the k vertices before this vertex in the path. Therefore, exactly one of the first $k + 1$ vertices has to be dominating. Let p_n^k be the independent domination polynomial of the k-path with n vertices. Then

$$p_n^k = x \sum_{i=1}^{k+1} p_{n-k-i}^k.$$

Formally, we define that $p_n^k = 1$ for all $n \leq 0$. As a consequence of the previous equation, we obtain a recurrence equation for the independent domination polynomial of the path.

Corollary 4.58. *Let $G = (V, E)$ be the path P_n with at least four vertices. Then*

$$\mathrm{D}_i(P_n, x) = x\,\mathrm{D}_i(P_{n-2}, x) + x\,\mathrm{D}_i(P_{n-3}, x).$$

The initial conditions are

$$\mathrm{D}_i(P_1, x) = x,$$
$$\mathrm{D}_i(P_2, x) = 2x \text{ and}$$
$$\mathrm{D}_i(P_3, x) = x^2 + x.$$

We can use the standard method to solve a recurrence equation and obtain

$$G(z) = \frac{xz + 2xz^2 + xz^3}{1 - xz^2 - xz^3}.$$

If x is equal one, this function simplifies to

$$G(z) = -\frac{z^3 + 2z^2 + z}{z^3 + z^2 - 1}.$$

The series $a(n)$, listed as A000931 in the On-Line Encyclopedia of Integer Sequences [OEI14], counts the number of compositions of n in parts that are odd and greater than or equal to three. Moreover $a(n)$ is also the number of strings of length $n-8$ from an alphabet $\{A, B\}$ with no more than one A or two B's consecutively. Now taking the generating function of the series A000931

$$H(z) = \frac{1 - z^2}{1 - z^2 - z^3}$$

and shifting the series by six yields

$$\frac{1}{z^6}\left(\frac{1 - z^2}{1 - z^2 - z^3} - 1 - z^3 - z^5 - z^6\right) = -\frac{z^3 + 2z^2 + z}{z^3 + z^2 - 1}.$$

This result is equal to the generating function $G(z)$. Therefore, we can use the explicit formula for $a(n+3)$ posted by Paul Barry [OEI14]

$$a(n+3) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{k}{n-2k}$$

to obtain a formula for the number of independent dominating sets of the path P_n:

$$D_i(P_n, 1) = \sum_{k=1}^{\lfloor (n+3)/2 \rfloor} \binom{k}{n-2k+3}.$$

There is also a more explicit formula for $a(n)$ given by Keith Schneider [OEI14]. Using this formula, we obtain:

$$D_i(P_n, 1) = \frac{r^{n+6}}{2r+3} + \frac{s^{n+6}}{2s+3} + \frac{t^{n+6}}{2t+3}, \tag{4.8}$$

where r, s, t are the three roots of $x^3 - x - 1$.

Moreover, we can use these ideas to obtain an explicit formula for the independent domination polynomial of the path P_n:

$$D_i(P_n, x) = \sum_{k=1}^{\lfloor (n+3)/2 \rfloor} \binom{k+1}{n-2k+1} x^k. \tag{4.9}$$

Because of the fact that the binomial coefficient is zero for some of the k's, we can restrict the range of the summation. The two inequalities

$$k + 1 \geq n - 2k + 1 \Leftrightarrow k \geq \frac{n}{3} \text{ and}$$
$$n - 2k + 1 \geq 0 \Leftrightarrow k \leq \frac{n+1}{2}$$

lead to

$$D_i(P_n, x) = \sum_{k=\lceil n/3 \rceil}^{\lfloor (n+1)/2 \rfloor} \binom{k+1}{n-2k+1} x^k.$$

We can use the polynomial of the path P_n to prove a theorem for the cycle C_n.

Theorem 4.59. *Let $G = (V, E)$ be the cycle C_n $(n \geq 5)$. Then*

$$D_i(C_n, x) = 2x\, D_i(P_{n-3}, x) + x^2\, D_i(P_{n-6}, x).$$

Proof. We number the vertices of the cycle as $1, \ldots, n$. If the vertex 1 of the cycle is dominating, then its two neighbors 2 and n are dominated and they cannot be dominating. This case will be counted by $x \, \mathrm{D}_i(P_{n-3}, x)$. If the vertex 1 is not dominating, then one of its neighbors must be dominating. If the vertex 2 is dominating, then the first vertex is dominated and the vertex n can either be dominating or not. This case will be counted by $x \, \mathrm{D}_i(P_{n-3}, x)$. If the vertices 1 and 2 are non-dominating, then the vertices 3 and n must be dominating. This yields the last part of the sum and the theorem is proved. $\qquad \square$

Using Equation (4.9) yields

$$\mathrm{D}_i(C_n, x) = \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \left(2 \binom{k+2}{n-2k-4} + \binom{k+1}{n-2k-5} \right) x^{k+2}.$$

Lemma 4.60. *Let $G = (V, E)$ be a wheel graph W_n with n vertices ($n \geq 4$). Then*

$$\mathrm{D}_i(W_n, x) = x + \mathrm{D}_i(C_{n-1}, x).$$

Proof. If the center vertex is dominating, then all other vertices are dominated. If the center vertex is non-dominating, then every independent dominating set of the vertices of the cycle is an independent dominating set of the whole graph. $\qquad \square$

Lemma 4.61. *Let $G = (V, E)$ be a fan graph F_n with n vertices ($n \geq 3$). Then*

$$\mathrm{D}_i(F_n, x) = x + \mathrm{D}_i(P_{n-1}, x).$$

Proof. The proof uses the same argumentation as the proof of Lemma 4.60. $\qquad \square$

4.4.1 Trees

First we investigate some special trees, which are well known in the literature and have some nice properties. In this section, also an algorithm for arbitrary trees will be given.

The *centipede* Cen_n was introduced by Levit and Mandrescu [LM05] and it is a tree obtained by the union of a path P_n and the edgeless graph E_n together with n edges connecting every vertex of the path with exactly one vertex of the E_n and vice versa (see Figure 4.8). Levit and Mandrescu proved a recurrence equation for the independence polynomial and it is already known [Weib] that the rank polynomial of the centipede is

$$\mathrm{R}(\mathrm{Cen}_n, x, y) = (1 + x)^{2n}.$$
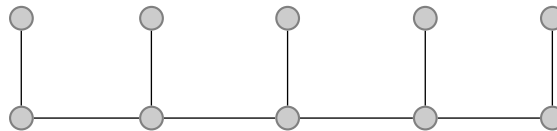

Fig. 4.8: The Centipede Cen_5.

If the first vertex of the path is dominating, then it dominates its adjacent pendant vertex and the second vertex of the path. Therefore, the adjacent pendant vertex of the second

vertex must be dominating. If the first vertex of the path is non-dominating, then its adjacent pendant vertex must be dominating. These two cases together yields

$$\mathrm{D}_i(\mathrm{Cen}_n, x) = x\,\mathrm{D}_i(\mathrm{Cen}_{n-1}, x) + x^2\,\mathrm{D}_i(\mathrm{Cen}_{n-2}, x)$$

with the initial conditions $\mathrm{D}_i(\mathrm{Cen}_0, x) = 1$ and $\mathrm{D}_i(\mathrm{Cen}_1, x) = 2x$.

Solving this recurrence equation yields

$$\mathrm{D}_i(\mathrm{Cen}_n, x) = \frac{(5 - 3\sqrt{5})(1 - \sqrt{5})^n + (5 + 3\sqrt{5})(1 + \sqrt{5})^n}{2^{n+1}5} x^n.$$

Please note that $\mathrm{D}_i(\mathrm{Cen}_n, 1)$ is the number of binary sequences of length n that have no consecutive zeros. This sequence is the well known Fibonacci sequence (for more information see the series A000045 in the On-Line Encyclopedia of Integer Sequences [OEI14]).

In a certain sense, a generalization of the centipede is the firecracker. An (n, k)-*firecracker* $F_{n,k}$, $n, k \geq 2$, is a graph obtained by the concatenation of n stars S_k by linking one leaf from each, such that the linked leaves form a (induced) path P_n in $F_{n,k}$ (see Figure 4.9). If k is equal to two, then $F_{n,2} \cong \mathrm{Cen}_n$. Please note that the independent domination polynomial of the star S_k is $x + x^{k-1}$.
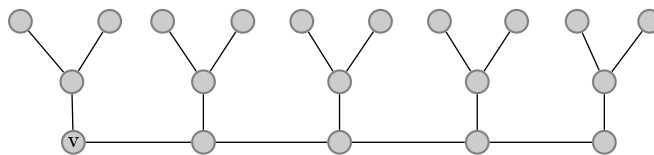
Fig. 4.9: Firecracker $F_{5,4}$.

Theorem 4.62. *Let $F_{n,k}$ be a (n, k)-firecracker with $n \geq 2$ and $k \geq 3$. Then*

$$\mathrm{D}_i(F_{n,k}, x) = (x^{2k-3} + x^k)\,\mathrm{D}_i(F_{n-2,k}, x) + x\,\mathrm{D}_i(F_{n-1,k}, x) + (x^{3k-5} + x^{2k-2})\,\mathrm{D}_i(F_{n-3,k}, x),$$

with the initial conditions

$$\begin{aligned}
\mathrm{D}_i(F_{0,k}, x) &= 1, \\
\mathrm{D}_i(F_{1,k}, x) &= x + x^{k-1} \text{ and} \\
\mathrm{D}_i(F_{2,k}, x) &= 2x^{2k-3} + 2x^k + x^2.
\end{aligned}$$

Proof. Suppose that the first vertex in the induced P_n is dominating (the vertex v in Figure 4.9), then the center vertex of the corresponding star is dominated and the remaining $k - 2$ vertices must be dominating. In the second star one leaf is dominated and therefore we have to multiply the polynomial of this case with $\mathrm{D}_i(S_{k-1}, x) = x + x^{k-2}$. This yields the first part of the sum in the theorem.

If the first vertex of the path is non-dominating, then it either can be dominated by the center vertex of the corresponding star or only by the second vertex of the path. In the first case, the center vertex has to be dominating and all vertices of the first star are dominated. In the second case, the second vertex in the path dominates the first and the third vertex and the center vertex of the second star. Therefore, we have two non-dominated stars with $k - 1$ vertices and $k - 2$ isolated vertices, together with the rest of the firecracker. This yields the last part of the sum of the theorem.   □

An (n, k)-banana tree $B_{n,k}$ is a graph obtained by connecting one leaf of each of the n copies of a star S_k with a single root vertex which is distinct from all the stars (see Figure 4.10). For k equal to one the $(n, 1)$-banana tree is isomorphic to the star S_{n+1}. The rank polynomial of the (n, k)-banana tree is [Weia]

$$\mathrm{R}(B_{n,k}; x, y) = (1 + x)^{nk}.$$
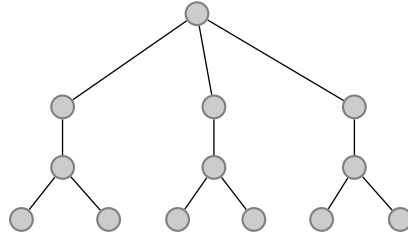
Fig. 4.10: Banana tree $B_{3,4}$.

Theorem 4.63. *Let $B_{n,k}$ be a (n, k)-banana tree, with $n \geq 1$ and $k \geq 2$. Then*

$$\mathrm{D}_i(\mathrm{B}_{n,k}, x) = x\,\mathrm{D}_i(\mathrm{S}_{k-1}, x)^n + \mathrm{D}_i(\mathrm{S}_k, x)^n - x^n.$$

Proof. If the root is dominating, then in every star one vertex is dominated and we have n remaining stars with $k - 1$ vertices. If the root is non-dominating, then we calculate the product of the independent domination polynomial of the n stars. But we also counted the case that the center vertex is dominating in all stars and therefore the root of the banana tree will not be dominated. Hence, we have to subtract the polynomial of this case and the theorem follows. $\qquad\square$

Like for other graph polynomials, the calculation of the independent domination polynomial of arbitrary trees can be done very fast. The first step is to transform the tree into a rooted tree and then calculate the vector P_u with three components for every vertex of the tree (see Section 3.2.5). The first component of P_u is the independent domination polynomial under the condition that the vertex u is dominating and all vertices lower than u are independently dominated. The second component of the vector is the polynomial under the condition that the vertex u is non-dominating, but u is dominated from a child. The last component is the polynomial under the condition that u is non-dominating and is not dominated from a child. The sum of the first two components of the P-vector of the root yields the independent domination polynomial of the tree.

It remains to specify the two operations of the tree algorithm (see Section 3.2.5). If we go one step upwards ($\oplus$), then we add the vertex w and the edge $\{u, w\}$. The vertex w can only be dominating if the vertex u is not dominating. Therefore, $P_w^1 = P_u^2 + P_u^3$. If the vertex w is dominated but non-dominating, then the vertex u must be dominating and hence $P_w^2 = P_u^1$. If the vertex w is non-dominating and non-dominated, then the vertex u must be dominated and therefore $P_w^3 = P_u^2$.

Let now w be a vertex with two child vertices u and v and let $\dot{P}_w$ be the vector for w obtained from u and $\ddot{P}_w$ the vector obtained from v. If the vertex w is a dominating vertex, then it must be a dominating vertex in both branches. If w is dominated, then at least one child must be a dominating vertex. If w is neither dominating nor dominated by another vertex, we must multiply the two possible cases.

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$$P_w = \dot{P}_w \otimes \ddot{P}_w = \begin{pmatrix} x\dot{P}_w^1\ddot{P}_w^1 \\ \dot{P}_w^2\ddot{P}_w^2 + \dot{P}_w^2\ddot{P}_w^3 + \dot{P}_w^3\ddot{P}_w^2 \\ \dot{P}_w^3\ddot{P}_w^3 \end{pmatrix}.$$

4.5 Independent Domination Reliability

Like for other counting problems, it makes sense to look for corresponding problems in the reliability context. Doing this, we find two different perspectives. The first point of view is that all vertices of the graph are dominating with a given probability p and the edges are perfectly reliable. On the other hand, we can assume that the edges of the graph are subject to random failure and we ask for the probability that an independent dominating set with (at most) k vertices exists. In this thesis, we only investigate the first case.

Definition 4.64. *Let $G = (V, E)$ be a graph whose vertices are subject to random and independent failure with probability $q = 1 - p$ and $|V| = n$. Then the* independent domination reliability polynomial *is defined as*

$$\mathrm{DRel}_i(G, p) = \sum_{k=1}^{n} d_k^i(G) p^k (1-p)^{n-k}.$$

According to this definition, the independent domination reliability polynomial is equivalent to the independent domination polynomial. With $q = 1 - p$ we obtain

$$\mathrm{DRel}_i(G, p) = q^n \mathrm{D}_i(G, p/q). \tag{4.10}$$

On the other hand we can obtain the independent domination polynomial from the reliability polynomial with

$$\mathrm{D}_i(G, x) = (1 + x)^n \mathrm{DRel}_i(G, x/(1 + x)). \tag{4.11}$$

The independent domination reliability function is not s-shaped as it is typical for many reliability functions, e.g. all-terminal reliability. Figure 4.11 shows the independent domination reliability polynomials of the diamond graph, the path P_5 and the Petersen graph. This shape is a consequence of the fact that if all vertices are dominating, then the resulting dominating set is not an independent set in the graph. Therefore, one interesting property is the maximum point of the function. More precisely, which value of p gives the highest independent domination reliability of the graph? The second interesting question in this context is whether it is possible to approximate the value of the independent domination reliability polynomial at $p = 0.5$. An approximation of this value together with Equation (4.11) yields an approximation for the number of independent dominating vertex subsets in the graph.

The first simple observation is that if the graph G is connected, then $\mathrm{DRel}_i(G, 0) = 0$ and $\mathrm{DRel}_i(G, 1) = 0$. Moreover, this is also correct if the graph G has at least one covered component.

Let now $\mathrm{DRel}_i(G, p)'$ be the first derivation of the independent domination reliability polynomial with respect to p.

Lemma 4.65. *Let K_n be the complete graph with n vertices and $\mathrm{DRel}_i(K_n, p)$ its independent domination reliability polynomial. Then $1/n$ is a root of the first derivation $\mathrm{DRel}_i(K_n, p)'$, $\mathrm{DRel}_i(K_n, 1/n) = \left(1 - \frac{1}{n}\right)^{n-1}$ and*

$$\lim_{n \to \infty} \mathrm{DRel}_i(K_n, 1/n) = e^{-1}.$$
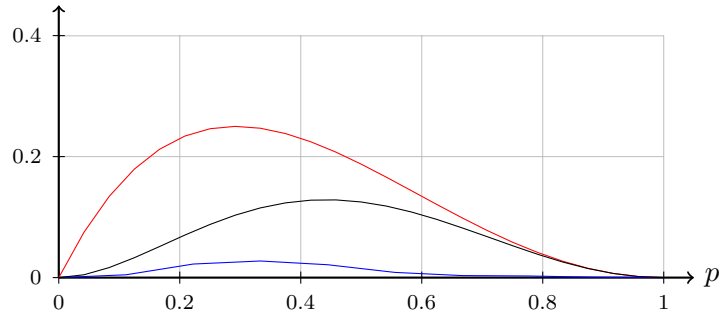
Fig. 4.11: Reliability functions of the diamond graph (red), the path P_5 (black) and the Petersen graph (blue).

Proof. Equation (4.7) together with Equation (4.10) yields

$$\text{DRel}_i(K_n, p) = np(1-p)^{n-1}.$$

If we randomly choose a vertex subset W of the K_n, such that p is the probability that a single vertex is in this set. Then the expected number of chosen vertices $E(|W|)$ is pn. In a complete graph, only the vertex subsets of size one are the independent dominating sets. The expected number of chosen vertices is equal to one if $p = 1/n$. Now inserting $p = 1/n$ in $\text{DRel}_i(K_n, p)$ results in the theorem. $\square$

In case of the complete bipartite graph the situation is more complex. For some complete bipartite graphs the reliability function has the expected shape, e.g. for the $K_{3,3}$ and the $K_{3,4}$. But if the sizes of the two bipartite sets are highly unbalanced, the reliability function has a local minimum at $p = 0.5$ and a maximum in each of the two intervals $(0, 0.5)$ and $(0.5, 1)$ (see Figure 4.12).
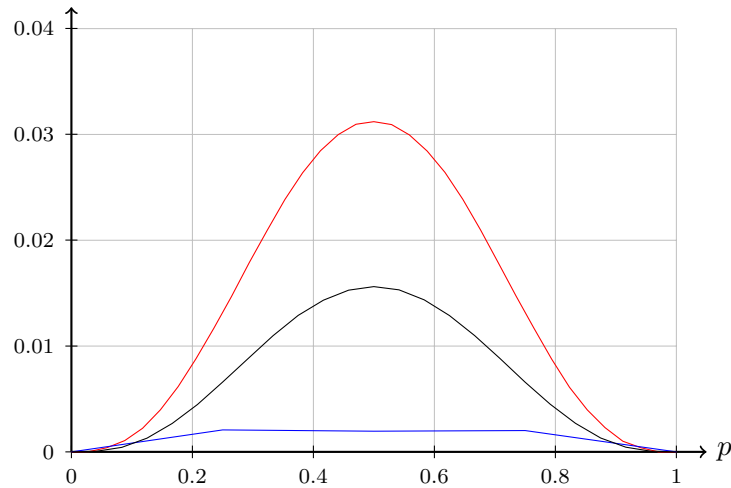
Fig. 4.12: Reliability functions of the $K_{3,3}$ (red), the $K_{3,4}$ (black) and the $K_{3,7}$ (blue).

Lemma 4.66. *The independent domination reliability polynomial* $\text{DRel}_i(K_{m,n}, p)$ *is symmetric at* $p = 0.5$ *in the interval* $[0, 1]$ *for all* $m, n \geq 2$.

Proof. The Theorem 4.56 together with Equation (4.10) yields

$$\text{DRel}_i(K_{m,n}, p) = p^n(1-p)^m + p^m(1-p)^n.$$

Inserting in the polynomial the two points $0.5 - r$ and $0.5 + r$, $r \in [0, 0.5]$, yields

$$(0.5 + r)^n (1 - (0.5 + r))^m + (0.5 + r)^m (1 - (0.5 + r))^n$$
$$= (0.5 + r)^n (0.5 - r)^m + (0.5 + r)^m (0.5 - r)^n$$

and

$$(0.5 - r)^n (1 - (0.5 - r))^m + (0.5 - r)^m (1 - (0.5 - r))^n$$
$$= (0.5 - r)^n (0.5 + r)^m + (0.5 - r)^m (0.5 + r)^n.$$

The two equations are equal and therefore the lemma follows. $\qquad \square$

Theorem 4.67. *The independent domination reliability polynomial* $\mathrm{DRel}_i(K_{m,n}, p)$ *has a maximum at* $p = 0.5$ *if and only if*

$$1 \le m < 3 \text{ and } 1 \le n \le \frac{1 + 2m}{2} + \frac{\sqrt{1 + 8m}}{2}$$

or

$$m \ge 3 \text{ and } \frac{1 + 2m}{2} - \frac{\sqrt{1 + 8m}}{2} \le n \le \frac{1 + 2m}{2} + \frac{\sqrt{1 + 8m}}{2}.$$

Proof. The first derivation of $\mathrm{DRel}_i(K_{m,n}, p)$ is

$$\frac{d}{dp} \mathrm{DRel}_i(K_{m,n}, p) = m(1 - p)^n p^{m-1} - m(1 - p)^{m-1} p^n - n(1 - p)^{n-1} p^m + n(1 - p)^m p^{n-1}$$

and the second derivation is

$$\frac{d}{dp^2} \mathrm{DRel}_i(K_{m,n}, p) = m(m - 1)(1 - p)^n p^{m-2} + m(m - 1)(1 - p)^{m-2} p^n$$
$$- 2nm(1 - p)^{n-1} p^{m-1} - 2nm(1 - p)^{m-1} p^{n-1}$$
$$+ n(n - 1)(1 - p)^{n-2} p^m + n(n - 1)(1 - p)^m p^{n-2}.$$

Inserting $p = 0.5$ in the first derivation yields $\mathrm{DRel}_i(K_{m,n}, 0.5)' = 0$ and in the second derivation resolves in

$$\mathrm{DRel}_i(K_{m,n}, 0.5)'' = \left(8m^2 - 16nm - 8m + 8n^2 - 8n\right) 2^{-m-n}.$$

This value can either be positive or negative, depending on the values of m and n. Solving the inequality $\mathrm{DRel}_i(K_{m,n}, 0.5)'' < 0$ leads to the theorem. $\qquad \square$

The star graph S_n is a special complete bipartite graph $K_{1,n-1}$ and therefore the statements about the symmetry and the extremal points at $p = 0.5$ are valid.

Corollary 4.68. *Let* $\mathrm{DRel}_i(S_n, p)$ *be the independent domination reliability polynomial of the star* S_n. *Then this polynomial is symmetric at* $p = 0.5$ *in the interval* $[0, 1]$, *for all* $n \ge 2$, *and it has its maximum at* $p = 0.5$ *if* $2 \le n \le 4$.

5 The Total Domination Polynomial

The concept of total domination in graphs was introduced by Cockayne, Dawes and Hedetniemi in 1980 [CDH80]. Some results about the total dominating set of a graph can be found in the two books of Haynes, Hedetniemi and Slater [HHS98b; HHS98a] and in a survey of Henning [Hen09]. Pfaff et al. [PLH83] showed that the decision problem whether an arbitrary graph has a total dominating set of a given size is NP-complete, even for bipartite graphs. The total domination polynomial was first introduced by Vijayan and Kumar [VK12b] in 2012. But in literature only some partial results about the total domination polynomial of cycles and paths are known [VK12a; VK12c].

Definition 5.1. *A vertex subset $W \subseteq V$ is called a* total dominating set *if $N(v) \cap W \neq \emptyset$ for all $v \in V$.*

It is also possible to define the total dominating sets with the total open neighborhood of a vertex subset. A vertex subset $W \subseteq V$ is total dominating if $N_G^t(W) = V$. We denote with $d_t(G)$ the number of the total dominating sets of the graph G. Now it is possible to define the total domination polynomial of a graph, which is the ordinary generating function for the number of total dominating sets in a graph.

Definition 5.2. *Let $G = (V, E)$ be a simple graph and $d_k^t(G)$ be the number of total dominating sets in G of size k. Then the* total domination polynomial *is defined as follows*

$$D_t(G, x) = \sum_{k=2}^{n} d_k^t(G) x^k.$$

Remark 5.3. *Every total dominating set is a dominating set, but not every dominating set is a total dominating set. So $d_k^t(G) \leq d_k(G)$.*

Like many other graph polynomials the total domination polynomial is multiplicative with respect to the components of the graph.

Theorem 5.4. *Let $G = (V, E)$ be a graph and $G_1, \ldots, G_k$ be the k components of the graph. Then*

$$D_t(G, x) = \prod_{i=1}^{k} D_t(G_i, x). \tag{5.1}$$

Proof. The theorem follows directly from the definition of the total domination polynomial. $\square$

Many results of the domination polynomial can be proved for the total domination polynomial. Kotek, Preen and Tittmann [KPT13] proved that the sum over the domination polynomials of all vertex induced subgraphs of a graph is equal $1 + (-x)^{|V|}$. A similar result also holds for the total domination polynomial:

Theorem 5.5. *Let $G = (V, E)$ be a connected graph with at least two vertices. Then*

$$\sum_{W \subseteq V} (-1)^{|W|} D_t(G[W], x) = 1 + (-x)^{|V|}.$$

Proof. The proof follows from the proof of Theorem 4.8. First of all, we insert the definition of the total domination polynomial in the equation and change the order of the summation. The argumentation of the single steps is the same as in the corresponding theorem for the independent domination polynomial, except for the usage of the total open neighborhood.

$$
\begin{aligned}
\sum_{W \subseteq V} (-1)^{|W|} D_t(G[W], x) &= \sum_{W \subseteq V} (-1)^{|W|} \sum_{\substack{U \subseteq W \\ N_{G[W]}^t(U) = W}} x^{|U|} \\
&= \sum_{U \subseteq V} x^{|U|} \sum_{\substack{W : U \subseteq W \\ N_{G[W]}^t(U) = W}} (-1)^{|W|} \\
&= \sum_{U \subseteq V} x^{|U|} \sum_{W : U \subseteq W \subseteq N_{G[W]}^t(U)} (-1)^{|W|} \\
&= \sum_{\substack{U \subseteq V \\ U \subseteq N_G^t(U)}} x^{|U|} \sum_{W : U \subseteq W \subseteq N_G[U]} (-1)^{|W|} \\
&= \sum_{\substack{U \subseteq V \\ U \subseteq N_G^t(U)}} (-x)^{|U|} \sum_{Y \subseteq N_G(U)} (-1)^{|Y|} \\
&= 1 + (-x)^{|V|}.
\end{aligned}
$$

$\square$

The last theorem together with the type of the graph (see Definition 2.8) yields the next corollary.

Corollary 5.6. *Let $G = (V, E)$ be a graph. Then*

$$\sum_{W \subseteq V} (-1)^{|W|} D_t(G[W], x) = \prod_{\substack{i \in \lambda_G \\ i \neq 1}} \left(1 + (-x)^i\right).$$

Proof. If G has an isolated vertex, e.g. v, then all terms in the sum are equal to zero if $v \in W$. Therefore, we only need to sum over $V \setminus \{v\}$ and obtain the corollary. $\square$

Applying the Möbius inversion to Corollary 5.6 yields the following equation.

$$D_t(G, x) = \sum_{W \subseteq V} (-1)^{|W|} \prod_{\substack{i \in \lambda_{G[W]} \\ i \neq 1}} \left(1 + (-x)^i\right).$$

We call a graph G *conformal* if all of its components are either of order one or of even order. Let $\mathrm{Con}(G)$ be the set of all vertex-induced conformal subgraphs of G.

Corollary 5.7. *Let $G = (V, E)$ be a graph. Then*

$$\mathrm{d}_t(G) = \sum_{H \in \mathrm{Con}(G)} (-1)^{\mathrm{iso}(H)} 2^{c(H)}.$$

Corollary 5.7 shows that, in contrast to the number of domination sets, the number of total dominating sets can either be even or odd.

The next theorem can be proved with the principle of inclusion-exclusion. We simply use a result about the probabilistic version of the total domination polynomial. Therefore, this theorem is a direct result of Theorem 5.35.

Theorem 5.8. *Let $G = (V, E)$ be a graph. Then*

$$\mathrm{D}_t(G, x) = \sum_{W \subseteq V} (-1)^{|W|}(1 + x)^{|V \setminus N_G^t(W)|}.$$

Proof. Applying Corollary 5.34 to Theorem 5.35 yields

$$\begin{aligned}
\mathrm{D}_t(G, x) &= (1 + x)^{|V|} \, \mathrm{DRel}_t(G, \frac{x}{x+1}) \\
&= (1 + x)^{|V|} \sum_{W \subseteq V} (-1)^{|W|}(1 - \frac{x}{1 + x})^{|N_G^t(W)|} \\
&= (1 + x)^{|V|} \sum_{W \subseteq V} (-1)^{|W|}(1 + x)^{-|N_G^t(W)|} \\
&= \sum_{W \subseteq V} (-1)^{|W|}(1 + x)^{|V \setminus N_G^t(W)|}.
\end{aligned}$$

□

Corollary 5.9. *Let $G = (V, E)$ be a graph with n vertices. Then the total domination polynomial satisfies*

$$\mathrm{D}_t(G, x) = \sum_{k=0}^{n} x^k \sum_{\substack{W \subseteq V \\ |N_G^t(W)| \leq n-k}} (-1)^{|W|} \binom{n - |N_G^t(W)|}{k}.$$

Proof. The proof follows exactly the proof in [KPT13], except for using the total open neighborhood instead of the closed neighborhood. Using Theorem 5.8, we obtain

$$\begin{aligned}
\mathrm{D}_t(G, x) &= \sum_{W \subseteq V} (-1)^{|W|}(1 + x)^{|V \setminus N_G^t(W)|} \\
&= \sum_{W \subseteq V} (-1)^{|W|} \sum_{k=0}^{n - |N_G^t(W)|} \binom{n - |N_G^t(W)|}{k} x^k \\
&= \sum_{k=0}^{n} x^k \sum_{W \subseteq V} (-1)^{|W|} \binom{n - |N_G^t(W)|}{k} \\
&= \sum_{k=0}^{n} x^k \sum_{\substack{W \subseteq V \\ |N_G^t(W)| \leq n-k}} (-1)^{|W|} \binom{n - |N_G^t(W)|}{k}.
\end{aligned}$$

□

5.1 On the t-Essential Sets of a Graph

The essential sets of a graph were introduced by Kotek et al. [KPT13]. They showed that only the essential sets of the graph are necessary for the calculation of the domination polynomial. We introduce the t-essential sets of the graph G and prove a similar result for the total domination polynomial.

Definition 5.10. *Let $G = (V, E)$ be a graph and W be a vertex subset of the graph. The set W is called* t-essential *if W contains the open neighborhood of at least one vertex of the graph. We denote the family of t-essential sets of G by $\mathrm{Ess}_t(G)$, in formula:*

$$\mathrm{Ess}_t(G) = \{X \subseteq V : \exists v \in V : X \supseteq N(v)\}.$$

Theorem 5.11. *Let $G = (V, E)$ be a graph and $n = |V|$. Then*

$$\mathrm{D}_t(G, x) = (-1)^n \sum_{W \in \mathrm{Ess}_t(G)} (-1)^{|W|} \left[(1 + x)^{|\{u \in V : N(u) \subseteq W\}|} - 1 \right].$$

Proof. Again using Theorem 5.8, we obtain

$$\mathrm{D}_t(G, x) = \sum_{W \subseteq V} (-1)^{|W|} (1 + x)^{|V \setminus N_G^t(W)|}$$
$$= \sum_{U \subseteq V} (-1)^{|V \setminus U|} (1 + x)^{|V \setminus N_G^t(V \setminus U)|}.$$

Now we investigate the exponent of $(1 + x)$:

$$V \setminus N_G^t(V \setminus U) = V \setminus \bigcup_{v \in V \setminus U} N(v)$$
$$= V \setminus \{u \in V : N(u) \cap (V \setminus U) \neq \emptyset\}$$
$$= V \setminus \{u \in V : N(u) \nsubseteq U\}$$
$$= \{u \in V : N(u) \subseteq U\}.$$

So we obtain:
$$\mathrm{D}_t(G, x) = \sum_{U \subseteq V} (-1)^{|V \setminus U|} (1 + x)^{|\{u \in V : N(u) \subseteq U\}|}.$$

All polynomials $(1+x)^{|\{u \in V : N(u) \subseteq U\}|}$ have the constant term 1 and if $V \neq \emptyset$, then the constant term vanishes in the sum. Hence, we can write:

$$\mathrm{D}_t(G, x) = \sum_{U \subseteq V} (-1)^{|V \setminus U|} \left[(1 + x)^{|\{u \in V : N(u) \subseteq U\}|} - 1 \right].$$

If U is not a t-essential set, then $\{u \in V : N(u) \subseteq U\} = \emptyset$. Consequently, only the terms corresponding to t-essential sets are not vanishing and we can restrict the summation to the set of t-essential sets of the graph. $\qquad\square$

Theorem 5.11 can be used for the fast calculation of the total domination polynomial in graphs with a high minimum degree. But a fast generation of the t-essential sets is necessary.

Lemma 5.12. *Let $G = (V, E)$ be a graph. Then*

$$\min_{W \in \mathrm{Ess}_t(G)} \{|W|\} = \delta(G)$$

and

$$N(v) \in \mathrm{Ess}_t(G), v \in V.$$

Proof. The argumentation is the same as in the proof of Lemma 4.15.   □

5.2 Recurrence Equations

T. Kotek et al. presented in [Kot+12] some recurrence equations for the domination polynomial. These results can also be proved with some minor changes for the total domination polynomial.

Definition 5.13. *Let $G = (V, E)$ be a graph and $u \in V$. Then $p_u(G)$ is the total domination polynomial of $G - N[u]$ under the condition that all vertices in $N(u)$ will be dominated by a vertex of $G - N[u]$.*

Theorem 5.14. *Let $G = (V, E)$ be a graph. For any vertex $u \in V$ we obtain*

$$\mathrm{D}_t(G, x) = \mathrm{D}_t(G - u, x) + x \, \mathrm{D}_t(G/u, x) - (1 + x)p_u(G) + x^2 \sum_{v \in N(u)} \mathrm{D}_t(G - N[\{u, v\}], x).$$

Proof. If the vertex $u \in V$ is non-dominating, then at least one of its neighbors must be a dominating vertex. $\mathrm{D}_t(G - u, x)$ counts all total dominating sets in $G - u$, especially those total dominating sets W with $N(u) \cap W = \emptyset$. So we must subtract $p_u(G)$ and obtain the polynomial for the desired case. If u is a dominating vertex, then also at least one vertex in $N(u)$ must be a dominating vertex. The polynomial $\mathrm{D}_t(G/u, x) - p_u(G)$ counts the total dominating sets in $G - u$ with at least one dominating vertex in $N(u)$. But it is possible that a single vertex in $N(u)$, e.g. v is dominating, but no other vertex in $N(v)$ except of u. In G this dominating set is a total dominating set, but in $G - u$ it is only a dominating set and will not be counted by the previous term. The sum $x^2 \sum_{v \in N(u)} \mathrm{D}_t(G - N[u] - N[v], x)$ counts exactly these total dominating sets and the theorem follows.   □

Corollary 5.15. *Let $G = (V, E)$ be a graph and $u, v \in V$ be two non-adjacent vertices of the graph with $N(v) \subseteq N(u)$. Then*

$$\mathrm{D}_t(G, x) = \mathrm{D}_t(G - u, x) + x \, \mathrm{D}_t(G/u, x) + x^2 \sum_{w \in N(u) \cap N(v)} \mathrm{D}_t(G - N[\{u, w\}], x).$$

Proof. If $N(v) \subseteq N(u)$, then the vertex v has degree zero in $G - N[u]$ and therefore $p_u(G) = 0$. Let the vertex u and a vertex $w \in N(u) - N(v)$ be dominating. Then $v \in V \setminus N[\{u, w\}]$ is an isolated vertex in $G - N[\{u, w\}]$ and therefore $\mathrm{D}_t(G - N[\{u, w\}], x) = 0$. The remaining sum follows from Theorem 5.14.   □

Corollary 5.16. *Let $G = (V, E)$ be a graph and $u, v \in V$ be two vertices with $N[v] \subseteq N[u]$. Then*

$$\mathrm{D}_t(G, x) = \mathrm{D}_t(G - u, x) + x \, \mathrm{D}_t(G/u, x) + x^2 \sum_{w \in N(u)} \mathrm{D}_t(G - N[\{u, w\}], x).$$

Proof. If $N[v] \subseteq N[u]$, then v is only adjacent to u and some of its neighbors and cannot be dominated by a vertex in $G - N[u]$. Therefore, $p_u(G) = 0$. $\square$

Corollary 5.17. *Let $G = (V, E)$ be a graph, $u, v \in V$ be two vertices of the graph with $N[v] \subseteq N[u]$ and let $N(u)$ induce a clique in the graph. Then*

$$\mathrm{D}_t(G, x) = (1 + x)\, \mathrm{D}_t(G - u, x) + x^2 \sum_{w \in N(u)} \mathrm{D}_t(G - N[\{u, w\}], x).$$

Definition 5.18. *Let $G = (V, E)$ be a graph and $u, v \in V$. Then $p_{u,v}(G)$ is the generating function for the total dominating sets W in $G - u$ with $W \cap N(u) = \{v\}$.*

Lemma 5.19. *[Kot+12] Let $G = (V, E)$ be a graph and let $e = \{u, v\} \in E$. Then*

$$p_u(G - e) = p_{u,v}(G) + p_u(G).$$

Theorem 5.20. *Let $G = (V, E)$ be a graph and $e = \{u, v\} \in E$. Then*

$$
\begin{aligned}
\mathrm{D}_t(G, x) =\ & \mathrm{D}_t(G - e, x) + p_u(G - e) + p_v(G - e) - p_u(G) - p_v(G) \\
& + x^2 \Big(\mathrm{D}_t(G - N[\{u, v\}], x) \\
& + \mathrm{D}_t(G - (N[u] \backslash \{v\}), x) - p_v(G - N[u]) + x \sum_{w \in N(v)} \mathrm{D}_t(G - N[\{u, v, w\}], x) \\
& + \mathrm{D}_t(G - (N[v] \backslash \{u\}), x) - p_u(G - N[v]) + x \sum_{w \in N(u)} \mathrm{D}_t(G - N[\{u, v, w\}], x) \Big).
\end{aligned}
$$

Proof. The polynomial $\mathrm{D}_t(G, x) - \mathrm{D}_t(G - e, x)$ counts exactly those total dominating sets W of G which are not total dominating sets in $G - e$. There are two possible situations in which such total dominating sets occur. In Figure 5.1 and 5.2 the two possible cases are shown.
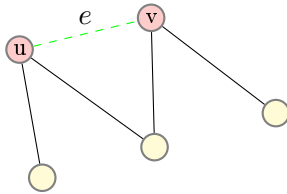
Fig. **5.1**: The vertices u and v are dominating.
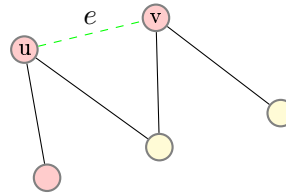
Fig. **5.2**: The vertices u, v and at least one vertex in $N(u) \backslash N(v)$ are dominating.

In the first case, the two end vertices u and v of the edge e are dominating and no vertex in the neighborhood of one of the two vertices is dominating. If no vertex in $N(\{u, v\})$ is dominating, then $x^2 \, \mathrm{D}_t(G - N[\{u, v\}], x)$ is the generating function for these total dominating sets (see Figure 5.1). Let now u and v be dominating vertices and u the only dominating vertex in the neighborhood of v and at least one vertex in the neighborhood of u, except of v, is dominating (see Figure 5.2). Then we can remove v and all vertices in the neighborhood of v except of u, because they will be dominated from v. Then we obtain the remaining part of the term with the argumentations of the second part of the proof of Theorem 5.14.

In the second case, one of the two vertices u and v is dominating and the other one is only dominated by this vertex. Precisely, if u is non-dominating, then the only neighbor of u which is dominating is v (see Figure 5.3). The function $p_{u,v}(G, x)$ counts exactly these total dominating sets. We also obtain the equivalent term for the other case. Together with Lemma 5.19 we obtain the theorem. $\square$
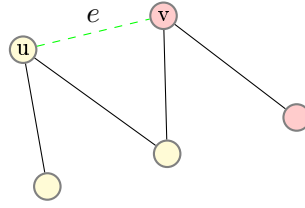
Fig. 5.3: Only the vertex v is dominating.

Corollary 5.21. *Let $G = (V, E)$ be a graph, $e = \{u, v\} \in E$ and $N[u] = N[v]$. Then*

$$\mathrm{D}_t(G, x) = \mathrm{D}_t(G - e, x) + x^2 \mathrm{D}_t(G - N[u], x).$$

Proof. The only case in which a total dominating set in G is not a total dominating set in $G - e$ is if both vertices u and v are dominating and no vertex in $N(u)\backslash\{v\}$ is dominating. These total dominating sets are counted by $x^2 \mathrm{D}_t(G - N[u], x)$. If only one vertex of $\{u, v\}$ is dominating, then a vertex in $N(u)\backslash\{v\}$ must be dominating. But if such a vertex exists, then this total dominating set is also a total dominating set in $G - e$. The same argumentation holds if neither u or v is dominating. $\qquad\square$

Theorem 5.22. *Let $G = (V, E)$ be a graph. Then*

$$\mathrm{D}_t(G, x) = \mathrm{D}_t(G - u, x) + \mathrm{D}_t(G \odot u, x) - \mathrm{D}_t(G \circledcirc u, x).$$

Proof. Applying Equation (2.1) to Theorem 5.14 yields

$$\begin{aligned}
\mathrm{D}_t(G, x) - \mathrm{D}_t(G - u, x) &= x \mathrm{D}_t(G/u, x) - (1 + x)p_u(G) \\
&\quad + x^2 \sum_{v \in N(u)} \mathrm{D}_t(G - N[\{u, v\}], x) \\
&= x \mathrm{D}_t((G \odot u)/u, x) - (1 + x)p_u(G \odot u) \\
&\quad + x^2 \sum_{v \in N(u)} \mathrm{D}_t(G - N[\{u, v\}], x).
\end{aligned} \tag{5.2}$$

Now apply Theorem 5.14 to the graph $G \odot u$

$$\begin{aligned}
\mathrm{D}_t(G \odot u, x) - \mathrm{D}_t((G \odot u) - u, x) &= x \mathrm{D}_t((G \odot u)/u, x) - (1 + x)p_u(G \odot u) \\
&\quad + x^2 \sum_{v \in N_{G\odot u}(u)} \mathrm{D}_t((G \odot u) - N_{G\odot u}[\{u, v\}], x). \tag{5.3}
\end{aligned}$$

Observe that $N_G(u) = N_{G\odot u}(u)$ and $(G \odot u) - N_{G\odot u}[\{u, v\}] = G - N_{G\odot u}[\{u, v\}]$. These two observations together with Equations (5.2) and (5.3) give the theorem. $\qquad\square$

5.3 Special Graph Classes

In this section we investigate the total domination polynomial in some special graph classes. First we prove theorems for complete and complete bipartite graphs.

Lemma 5.23. *Let $G = (V, E)$ be a complete graph with n vertices. Then*

$$\mathrm{D}_t(K_n, x) = (1 + x)^n - nx - 1.$$

Proof. In a complete graph every subset W of V, with $|W| > 1$, is a total dominating set. $\square$

Theorem 5.24. *Let* $G = (V, E)$ *be the complete bipartite graph* $K_{m,n} = (V_1 \cup V_2, E)$*. Then*

$$\mathrm{D}_t(K_{m,n}, x) = (1 + x)^{m+n} - (1 + x)^m - (1 + x)^n + 1.$$

Proof. Every vertex subset $W \subseteq V$, with $W \cap V_1 \neq \emptyset$ and $W \cap V_2 \neq \emptyset$, is a total dominating set. The term $(1+x)^{m+n}$ is the generating function for the subsets of V. So we must subtract the possibilities to choose subsets that consist only of vertices of V_1 or V_2, respectively. $\square$

Theorem 5.25. *Let* P_n *be a path with n vertices ($n \geq 5$) and* $p_n = \mathrm{D}_t(P_n, x)$ *its total domination polynomial. Then*

$$p_n = xp_{n-1} + x^2(p_{n-3} + p_{n-4}).$$

The initial conditions are

$$
\begin{aligned}
p_1 &= 0, \\
p_2 &= x^2, \\
p_3 &= x^3 + 2x^2, \\
p_4 &= x^4 + 2x^3 + x^2.
\end{aligned}
$$

Proof. Let p_n^1 be the total domination polynomial of the path P_n under the condition that the first vertex of the path is a dominating vertex and p_n^2 be the polynomial under the condition that the first vertex is non-dominating. Then we can write the total domination polynomial of the graph in the following way:

$$p_n = p_n^1 + p_n^2. \tag{5.4}$$

If the first vertex is non-dominating, then it must be dominated by the second vertex. This yields

$$p_n^2 = p_{n-1}^1. \tag{5.5}$$

If the first vertex of P_n is dominating, then the second one has to be dominating, too. We obtain two possibilities for the third vertex. If the third vertex is dominating, then it dominates the second vertex and this is counted by xp_{n-1}^1. If the third vertex is non-dominating, then it will be dominated by the second vertex. This leads to $x^2 p_{n-3}$. Together with Equation (5.5) we obtain

$$p_n^1 = xp_n^2 + x^2 p_{n-3}. \tag{5.6}$$

Inserting the Equations (5.5) and (5.6) in Equation (5.4) we obtain

$$
\begin{aligned}
p_n &= xp_n^2 + x^2 p_{n-3} + p_{n-1}^1 \\
&= x^2 p_{n-3} + xp_{n-1}^1 + (xp_{n-1}^2 + x^2 p_{n-4}) \\
&= xp_{n-1} + x^2(p_{n-3} + p_{n-4}).
\end{aligned}
$$

$\square$

To prove a theorem for the cycle we need the next two lemmas.

Lemma 5.26. *Let P_n be a path with n vertices ($n \geq 4$) and $p_n = \mathrm{D}_t(P_n, x)$. Let p_n^1 be the total domination polynomial of the path P_n if the first vertex is dominating. Then*

$$p_n^1 = x p_{n-1}^1 + x^2 p_{n-3}.$$

The initial conditions are

$$\begin{aligned}
p_1^1 &= x, \\
p_2^1 &= x^2, \\
p_3^1 &= x^2 + x^3.
\end{aligned}$$

Proof. If in the path P_n the first vertex is dominating, then the second vertex must be a dominating vertex. If the third vertex is also dominating, then we obtain $x p_{n-1}^1$. If the third vertex is non-dominating, then it will be dominated by the second vertex. In this case we obtain the polynomial $x^2 p_{n-3}$. □

Lemma 5.27. *Let P_n be a path with n vertices ($n \geq 7$) and $p_n = \mathrm{D}_t(P_n, x)$. Let p_n^3 be the total domination polynomial under the condition that both end vertices are dominating. Then*

$$p_n^3 = x^2 p_{n-2}^3 + 2x^3 p_{n-4}^1 + x^4 p_{n-6}.$$

The initial conditions are

$$\begin{aligned}
p_2^3 &= x^2, \quad p_3^3 = x^3, \\
p_4^3 &= x^4, \quad p_5^3 = x^4 + x^5, \\
p_6^3 &= x^4 (1 + x)^2.
\end{aligned}$$

Proof. If in the path P_n the two end vertices are dominating, then the second and the second to the last vertex must be dominating. Let now u be the third vertex and v be the last but two vertex. If both u and v are dominating vertices, then we can cut off the first vertex on each side and obtain $x^2 p_{n-2}^3$. If exactly one of the two vertices u and v is dominating (e.g. u), then the other vertex (e.g. v) will be dominated by its neighbor and has no influence on the other vertices of the graph. This yields $2x^3 p_{n-4}^1$. If both vertices are non-dominating, then they will be dominated by their neighbors and we can cut off the first three vertices of each side of the path and obtain $x^4 p_{n-6}$ for this case. □

The next two theorems give results for the cycle and the complement of a cycle.

Theorem 5.28. *Let $G = (V, E)$ be a cycle with n vertices ($n \geq 5$), then*

$$\mathrm{D}_t(C_n, x) = p_n^3 + p_{n-2}^3 + 2p_{n-1}^1 + 2x p_{n-2}^1 + x^2 p_{n-4}.$$

Proof. Let $u, v \in V$ be two vertices of the cycle with $\{u, v\} \in E$ and let u_1 be the other neighbor of u and v_1 the other neighbor of v. If u and v are dominating vertices, then u_1 and v_1 can be dominating or not. So we have three possible cases. If both vertices are dominating, then we can remove the edge e and obtain p_n^3. If neither u_1 nor v_1 are dominating, then they will be dominated from u, respectively v. This case is counted by $x^2 p_{n-4}$. If only one of the two vertices is dominating, e.g. u_1, then v_1 will be dominated by v and we can remove the edge e and obtain $x p_{n-2}^1$ for the remaining graph.

If only one of the two vertices u and v is a dominating vertex, e.g. u, then the next vertex on the cycle, here u_1, must also be a dominating vertex. On the other side v will be dominated
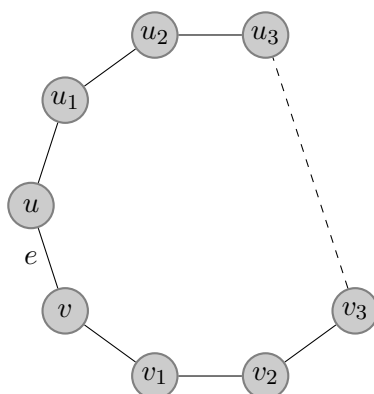
Fig. 5.4: Calculation of the total domination polynomial in a cycle.

by u and so we can delete the vertex v. With these considerations we obtain p_{n-1}^1 for this case. The last case is that neither u nor v are dominating vertices. Hence, u_1 and v_1 must be dominating vertices and we obtain p_{n-2}^3. $\qquad\square$

Theorem 5.29. *Let C_n be a cycle with at least five vertices. Then*

$$\mathrm{D}_t(\overline{C_n}, x) = (1+x)^n - n(x^3 + 2x^2 + x) - 1.$$

Proof. First note that every vertex subset $W \subseteq V$, with $|W| \geq 4$, is a total dominating set in $\overline{C_n}$. If we only choose three dominating vertices some subsets are not total dominating sets. These are exactly those sets where we choose three consecutive vertices. So we have $\binom{n}{3} - n$ possibilities to choose a total dominating set with three vertices. If we choose only two vertices, then all choices are valid except for the selection of two vertices that have the distance one or two in the cycle C_n. These are exactly $\binom{n}{2} - 2n$ possibilities. With these considerations we obtain

$$\begin{aligned}
\mathrm{D}_t(\overline{C_n}, x) &= \sum_{k=4}^n \binom{n}{k} x^k + \left(\binom{n}{3} - n\right) x^3 + \left(\binom{n}{2} - 2n\right) x^2 \\
&= \sum_{k=2}^n \binom{n}{k} x^k - nx^3 - 2nx^2 \\
&= (1+x)^n - n(x^3 + 2x^2 + x) - 1.
\end{aligned}$$

$\qquad\square$

We can use the previous results and prove the following theorem.

Theorem 5.30. *Let $G = (V, E)$ be a wheel graph W_n with n vertices ($n \geq 3$). Then*

$$\mathrm{D}_t(W_n, x) = \mathrm{D}_t(C_{n-1}, x) + x\left((1+x)^{n-1} - 1\right).$$

Proof. Let $v \in V$ be the center vertex of the wheel W_n. If v is not a dominating vertex, then the total domination polynomial of C_{n-1} is the polynomial for this case because every total dominating set in C_{n-1} dominates v. If v is a dominating vertex, then at least one vertex on the C_{n-1} must be dominating. This leads to the term

$$x\left((1+x)^{n-1} - 1\right).$$

$\qquad\square$

Corollary 5.31. *Let $G = (V, E)$ be a fan graph F_n with n vertices ($n \geq 3$), then from Theorem 5.30 follows:*

$$\mathrm{D}_t(F_n, x) = \mathrm{D}_t(P_{n-1}, x) + x\left((1 + x)^{n-1} - 1\right).$$

5.4 Total Domination Reliability Polynomial

Suppose we have a network of clients and every client controls the clients in its neighborhood. If the clients fail with a given probability, we can ask: What is the probability that every client will be controlled by another client? This question asks for the probability that there exists a total dominating set in the probabilistic graph. To answer this question, we define the total domination reliability polynomial.

Definition 5.32. *Let $G = (V, E)$ be a graph whose vertices are subject to random and independent failure with probability $q = 1 - p$. Then the* total domination reliability polynomial *$\mathrm{DRel}_t(G, p)$ is defined as follows*

$$\mathrm{DRel}_t(G, p) = \sum_{k=2}^{n} d_k^t(G)p^k(1 - p)^{n-k}.$$

The first lemma in this section shows the connection between the total domination reliability polynomial and the total domination polynomial.

Lemma 5.33. *Let $G = (V, E)$ be a graph and $\mathrm{D}_t(G, x)$ be the total domination polynomial of G. Then the total domination reliability polynomial can be calculated in the following way:*

$$\mathrm{DRel}_t(G, p) = (1 - p)^n \mathrm{D}_t(G, \frac{p}{1 - p}).$$

Corollary 5.34. *Let $G = (V, E)$ be a graph and $\mathrm{DRel}_t(G, p)$ the total domination reliability polynomial. Then*

$$\mathrm{D}_t(G, x) = (1 + x)^n \mathrm{DRel}_t(G, \frac{x}{x + 1}).$$

Theorem 5.35. *Let $G = (V, E)$ be a graph and the vertices of the graph are subject to random and independent failure with probability $q = 1 - p$. Then*

$$\mathrm{DRel}_t(G, p) = \sum_{W \subseteq V} (-1)^{|W|}(1 - p)^{|N_G^t(W)|}.$$

Proof. Let A_u be the event that no vertex in the open neighborhood of the vertex u is operating. Then $\bigcap_{u \in W} \overline{A_u}$ occurs if and only if the operating vertices form a total dominating set in G. The event $\bigcap_{u \in W} A_u$ occurs if and only if all vertices in $N_G^t(W)$ fail. This event happens with probability $q^{|N_G^t(W)|}$. Applying the inclusion-exclusion principle, the theorem follows. $\qquad\square$

5.5 The Trivariate Domination Polynomial

The enumeration of total dominating sets can be refined in the following way. We might want to distinguish all vertex subsets of a given graph with respect to its cardinality, the cardinality of its open neighborhood, and the number of isolated vertices in the induced subgraph, which yields a trivariate generating function.

Definition 5.36. *Let $G = (V, E)$ be a graph. Then the* trivariate domination polynomial *is given as follows:*

$$\mathrm{Y}(G; x, y, z) = \sum_{W \subseteq V} x^{|W|} y^{|N(W)|} z^{\mathrm{iso}(G[W])}.$$

Let $G = (V, E)$ be a graph and we denote by $t_{i,j,k}(G)$ the number of subsets W with $i = |W|$, $j = |N(W)|$ and $k = \mathrm{iso}(G[W])$. Then we can write the trivariate domination polynomial as

$$\mathrm{Y}(G; x, y, z) = \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{n} t_{i,j,k}(G) x^i y^j z^k.$$

Lemma 5.37. *Let $G = (V, E)$ be a graph with two components $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Then*

$$\mathrm{Y}(G; x, y, z) = \mathrm{Y}(G_1; x, y, z) \, \mathrm{Y}(G_2; x, y, z).$$

Proof. We can simply use the definition to prove the lemma:

$$\begin{aligned}
\mathrm{Y}(G; x, y, z) &= \sum_{W \subseteq V} x^{|W|} y^{|N(W)|} z^{\mathrm{iso}(G[W])} \\
&= \sum_{W \subseteq V_1 \cup V_2} x^{|W|} y^{|N(W)|} z^{\mathrm{iso}(G[W])} \\
&= \sum_{W_1 \subseteq V_1} x^{|W_1|} y^{|N(W_1)|} z^{\mathrm{iso}(G[W_1])} \sum_{W_2 \subseteq V_2} x^{|W_2|} y^{|N(W_2)|} z^{\mathrm{iso}(G[W_2])} \\
&= \mathrm{Y}(G_1; x, y, z) \, \mathrm{Y}(G_2; x, y, z).
\end{aligned}$$

$\square$

Theorem 5.38. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be graphs, with $|V(G)| = n$ and $|V(H)| = m$. Then the trivariate domination polynomial of the join of these two graphs can be calculated with*

$$\begin{aligned}
\mathrm{Y}(G * H; x, y, z) = {}& y^{n+m} \left[\left(\mathrm{Y}(G; \frac{x}{y}, 1, 1) - 1 \right) \left(\mathrm{Y}(H; \frac{x}{y}, 1, 1) - 1 \right) \right] \\
&+ y^m \left(\mathrm{Y}(G; x, y, z) - 1 \right) + y^n \left(\mathrm{Y}(H; x, y, z) - 1 \right) + 1.
\end{aligned}$$

Proof. If in both graphs at least one vertex is dominating, then all non-dominating vertices will be dominated and the dominating vertices induce a connected subgraph in $G * H$. This case will be counted by the first part of the sum. If in the graph H no vertex is dominating, but at least one vertex in G, then all vertices in H will be dominated. This is counted by $y^m (\mathrm{Y}(G; x, y, z) - 1)$. The same argumentation is valid if no vertex in G is dominating, but at least one in H. $\square$

Corollary 5.39. *Let $G = (V, E)$ be a graph and $|V| = n$. Then*

$$\mathrm{Y}(G * K_1; x, y, z) = xy^n \mathrm{Y}(G; \frac{x}{y}, 1, 1) + y\left(\mathrm{Y}(G; x, y, z) - 1\right)$$
$$+ (z - 1)xy^n + 1.$$

Proof. The new vertex $v \in V(K_1)$ is adjacent to all vertices in G. Therefore, if v is dominating, then all non-dominating vertices in G will be dominated by v. The term

$$xy^n \mathrm{Y}(G; \frac{x}{y}, 1, 1)$$

counts these cases. But if in G no vertex is dominating and v is dominating, then v is an isolated vertex in the induced subgraph. Hence, we must subtract xy^n and add $xy^n z$. If the vertex v is non-dominating, then it will be dominated from any dominating vertex in G. This case is counted by $y\left(\mathrm{Y}(G; x, y, z) - 1\right)$. $\qquad\square$

5.5.1 Encoded Graph Invariants

It is easy to verify that we can obtain the total domination polynomial from the trivariate domination polynomial with

$$\mathrm{D}_t(G, x) = [y^n]\,\mathrm{Y}(G; xy, y, 0),$$

the domination polynomial with

$$D(G, x) = [y^n]\,\mathrm{Y}(G; xy, y, 1)$$

and the independent domination polynomial with

$$\mathrm{D}_i(G, x) = [y^n z^n]\,\mathrm{Y}(G; xy, yz, z). \tag{5.7}$$

Theorem 5.40. *Let $G = (V, E)$ be a graph, $\mathrm{Y}(G; x, y, z)$ be the trivariate domination polynomial, $\mathrm{I}(G, x)$ be the independence polynomial and $\Psi(G, x)$ be the vertex-cover polynomial of the graph. Then*

$$\mathrm{I}(G, x) = \lim_{z \to \infty} \mathrm{Y}(G; \frac{x}{z}, 1, z) \text{ and}$$
$$\Psi(G, x) = x^{|V|} \lim_{z \to \infty} \mathrm{Y}(G; \frac{1}{xz}, 1, z)$$

Proof. A vertex subset W of the graph $G = (V, E)$ is an independent set if it only consists of isolated vertices in $G[W]$. Using the definition of the trivariate domination polynomial, we obtain:

$$\mathrm{Y}(G; \frac{x}{z}, 1, z) = \sum_{W \subseteq V} \left(\frac{x}{z}\right)^{|W|} z^{\mathrm{iso}(G[W])}$$
$$= \sum_{W \subseteq V} \frac{x^{|W|}}{z^{|W| - \mathrm{iso}(G[W])}}.$$

The term $|W| - \mathrm{iso}(G[W])$ is equal to zero if and only if W is an independent set. Otherwise, if W is not an independent set, then with the limes the corresponding summand vanishes. The connection to the vertex-cover polynomial follows from the connection between the independence polynomial and the vertex-cover polynomial. $\qquad\square$

In Figure 5.5 some connections between different domination related polynomials are shown. An arrow in the figure means that this graph polynomial can be obtained from the corresponding polynomial. A dashed arrow means that this connection only exists in some special graph classes.
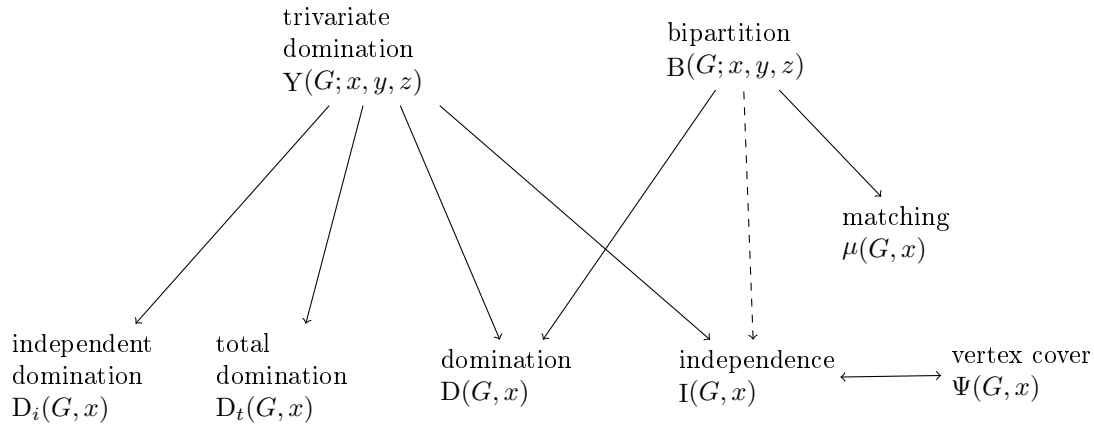
Fig. 5.5: Graph of domination related graph polynomials.

The next theorem shows some basic graph invariants which can be obtained from the trivariate domination polynomial.

Theorem 5.41. *Let $G = (V, E)$ be a graph and $Y(G; x, y, z)$ its trivariate domination polynomial. Let $k(G)$ be the number of the components of the graph and n_i be the order of the i-th component. Then*

$$|V| = \deg(Y(G; x, 1, 1)),$$

$$|E| = \frac{1}{2} \sum_{j=1}^{n-1} j \, t_{1,j,1} = [x^2 z^0] \, Y(G; x, 1, z),$$

$$\mathrm{iso}(G) = [x] \, Y(G; x, 0, 1),$$

$$Y(G; 1, 0, 1) = 2^{k(G)},$$

$$Y(G; x, 0, 1) = \prod_{i=1}^{k(G)} \left(1 + x^{n_i} \right).$$

Theorem 5.42. *Let $G = (V, E)$ be a graph. Then the degree generating function of G is*

$$\sum_{v \in V} t^{\deg(v)} = [x] \, Y(G; x, t, 1).$$

Proof. If we substitute y by t and z by 1, we obtain:

$$Y(G; x, t, 1) = \sum_{W \subseteq V} x^{|W|} t^{|N(W) \setminus W|}.$$

In the equation we only need those summands where the power of x is equal one. Therefore, in the sum we are only interested in subsets of size one. Hence, we obtain the degree generating function:

$$[x] \, Y(G; x, t, 1) = \sum_{v \in V} t^{|N(v)|}.$$

$\square$

5.5.2 Graph Products

In this section, we investigate the trivariate domination polynomial of some product graphs (see Section 2.3 for an introduction to the different product graphs).

Cartesian Product

The first product of interest is the Cartesian product of two graphs.

Theorem 5.43. *The trivariate domination polynomial of the Cartesian product of the complete graphs K_2 and K_n, with $n \geq 2$, can be calculated with*

$$\mathrm{Y}(K_2 \square K_n; x, y, z) = 2((y + xy)^n - y^n(1 + nx(1 - z))) + ((x + y)^n - y^n - nxy^{n-1})^2$$
$$+ 2nxy^n \left[\left(\frac{x}{y} + z \right) (x + y)^{n-1} - y^{n-1}z \right.$$
$$\left. + xy^{n-2} \left(\frac{1}{2}(n - 1)z^2 - (n - 1)z - \frac{1}{2} \right) \right].$$

Proof. To prove the theorem, we distinguish between three possible cases: 1. Only in one of the two rows there is a dominating vertex, 2. in both rows at least two vertices are dominating and 3. in one row exactly one vertex and in the other row at least one vertex is dominating. It is easy to see that the sum of the polynomials of these three cases yields the theorem.

1. Every dominating vertex dominates all other vertices in the same row and exactly one vertex in the other row. The polynomial $y^n(1 + yx/y)^n$ counts this case. But here we also count the choice of the empty set and if we have exactly one dominating vertex, then it is an isolated dominating vertex. This yields

$$2y^n((1 + x)^n - 1 - nx + nxz) = 2((y + xy)^n - y^n + nxy^n(z - 1))$$

for the first part of the proof.

2. If in both rows at least two vertices are dominating, then all other vertices are dominated and no isolated dominating vertex exists. This yields

$$y^{2n}((1 + x/y)^n - 1 - nx/y)^2.$$

3. If in the first row exactly one vertex is dominating, then it dominates all vertices in this row. If the adjacent vertex in the second row is non-dominating, then the dominating vertex is an isolated dominating vertex and we can choose dominating vertices in the remaining $(n-1)$ vertices. This yields:

$$nxy^{2n-1}z \left(\left(1 + \frac{x}{y} \right)^{n-1} - 1 + (n - 1)\frac{x}{y}(z - 1) \right). \tag{5.8}$$

If the vertex adjacent to the first vertex is dominating, then no isolated dominating vertex exists and we can choose dominating vertices from the remaining $(n-1)$ vertices. This yields:

$$nx^2y^{2n-2} \left(1 + \frac{x}{y} \right)^{n-1}. \tag{5.9}$$

The same argumentation holds for the reverse case. Therefore, we add (5.8) and (5.9) and multiply the result by two. But now we count the case that in any of both rows exactly one vertex is dominating twice. We count this case with

$$ny^{2n-1}x\left((n-1)\frac{x}{y}z^2+\frac{x}{y}\right)=ny^{2n-2}x^2\left((n-1)z^2+1\right).\qquad(5.10)$$

Summing the Equations (5.8), (5.9) and (5.10) yields:

$$\begin{aligned}
&2nxy^{2n-1}z\left(\left(1+\frac{x}{y}\right)^{n-1}-1+(n-1)\frac{x}{y}(z-1)\right)+2nx^2y^{2n-2}\left(1+\frac{x}{y}\right)^{n-1}\\
&\quad-ny^{2n-2}x^2\left((n-1)z^2+1\right)\\
&=2nxy^{2n-1}\left(z\left(1+\frac{x}{y}\right)^{n-1}-z-(n-1)\frac{x}{y}z+(n-1)\frac{x}{y}z^2\right.\\
&\qquad\left.+\frac{x}{y}\left(1+\frac{x}{y}\right)^{n-1}\right)-ny^{2n-2}x^2\left((n-1)z^2+1\right)\\
&=2nxy^n\left(z(y+x)^{n-1}-y^{n-1}z-(n-1)xy^{n-2}z+(n-1)xy^{n-2}z^2\right.\\
&\qquad\left.+\frac{x}{y}(y+x)^{n-1}\right)-ny^{2n-2}x^2\left((n-1)z^2+1\right)\\
&=2nxy^n\left(\left(z+\frac{x}{y}\right)(x+y)^{n-1}-y^{n-1}z+(n-1)xy^{n-2}z(z-1)\right.\\
&\qquad\left.-\frac{1}{2}xy^{n-2}((n-1)z^2+1)\right)\\
&=2nxy^n\left(\left(\frac{x}{y}+z\right)(x+y)^{n-1}-y^{n-1}z+xy^{n-2}\left(\frac{1}{2}(n-1)z^2-(n-1)z-\frac{1}{2}\right)\right).
\end{aligned}$$

Taking all cases together and doing some simplifications, we obtain the theorem. $\square$

Lexicographic Product

Theorem 5.44. *Let $G=(V,E)$ be a connected graph with m vertices ($m\geq 2$). Then the trivariate domination polynomial of the lexicographic product of the complete graph K_n with at least two vertices and the graph G can be calculated by*

$$\begin{aligned}
\mathrm{Y}(K_n\cdot G;x,y,z)=&\,ny^{(n-1)m}\left(\mathrm{Y}(G;x,y,z)-1\right)\\
&+y^{nm}\sum_{i=2}^{n}\binom{n}{i}\left((1+x/y)^m-1\right)^i+1.
\end{aligned}$$

Proof. The first observation is that a vertex is adjacent to all other vertices which are not in the same row. Therefore, if only in one row vertices are dominating, then all vertices outside this row and all adjacent vertices in this row are dominated. The polynomial $\mathrm{Y}(G;x,y,z)$ counts the vertex subsets of one row and therefore

$$ny^{(n-1)m}\left(\mathrm{Y}(G;x,y,z)-1\right)$$

is the polynomial for the first case.

If vertices in at least two rows are dominating, then all other vertices in the graph are dominated. This will be counted by

$$y^{nm} \sum_{i=2}^{n} \binom{n}{i} \left((1 + x/y)^m - 1\right)^i$$

and the theorem follows. $\qquad \square$

Theorem 5.45. *Let $G = (V, E)$ be a connected graph with at least two vertices. Then the trivariate domination polynomial of the lexicographic product of the graph G and the complete graph K_n $(n \geq 2)$ can be calculated by*

$$Y(G \cdot K_n; x, y, z) = Y\left(G; (y + x)^n - y^n, y^n, 1 + \frac{nx(z-1)/y}{(1 + x/y)^n - 1}\right).$$

Proof. Analog to the proof of Theorem 6.53. $\qquad \square$

Theorem 5.46. *Let $H = (V, E)$ be a connected graph with m vertices, $g_n = Y(P_n \cdot H; x, y, z)$ and $m, n \geq 2$. Furthermore, let f_n be the trivariate domination polynomial of $P_n \cdot H$ under the condition that in the first row at least one vertex is dominating. Then*

$$g_n = f_n + y^m \sum_{i=1}^{n-1} f_{n-i} + 1.$$

Proof. There are two possible cases with respect to the number of dominating vertices in the first row. The first case is that at least one vertex is dominating. This will be counted by f_n. If in the first row no vertex is dominating, but in the second row at least one vertex is dominating, then all vertices in the first row are dominated. This will be counted by $y^m f_{n-1}$. Repeating this process recursively yields the theorem. $\qquad \square$

For the recurrence equations in the next lemma we need the trivariate domination polynomials of the products $P_0 \cdot H$ and $P_1 \cdot H$, respectively. Without loss of generality we define $g_0 = 1$ and $g_1 = Y(H; x, y, z)$.

Lemma 5.47. *Let $H = (V, E)$ be a graph with m vertices, $g_n = Y(P_n \cdot H; x, y, z)$ and $m, n \geq 2$. Furthermore, let f_n be the trivariate domination polynomial of $P_n \cdot H$ under the condition that in the first row at least one vertex is dominating and h_n be the polynomial under the condition that the first row is already dominated and at least one vertex is dominating in it. Then*

$$f_n = (Y(H; x, y, z) - 1) y^m g_{n-2} + ((y + x)^m - y^m) h_{n-1} \text{ and}$$
$$h_n = ((y + x)^m - y^m) (y^m g_{n-2} + h_{n-1}).$$

The initial conditions are

$$f_1 = Y(H; x, y, z) - 1$$
$$h_1 = (y + x)^m - y^m.$$

Proof. First we calculate the polynomial under the condition that in the first row at least one vertex is dominating. For that purpose we distinguish two cases: (1) In the second row no vertex is dominating and (2) in the second row at least one vertex is dominating. If in the second row no vertex is dominating, but in the first one at least one, then all vertices in the second row are dominated. This case will be counted by $(\mathrm{Y}(H; x, y, z) - 1)\, y^m g_{n-2}$. But if in the second row at least one vertex is dominating, then all vertices in the first row are dominated independently from the dominating vertices in the first row. This yields $((y + x)^m - y^m)\, h_{n-1}$ and the first part of the theorem follows.

Suppose now that all vertices in the first row are dominated, but at least one vertex of this row must be dominating. In the first row we can choose any non-empty vertex subset and the non-dominating vertices are dominated. Again we distinguish between two cases with respect to the number of dominating vertices in the second row. If in the second row no vertex is dominating, then all vertices will be dominated and this case will be counted by $((y + x)^m - y^m)\, y^m g_{n-2}$. If in the second row at least one vertex is dominating, then we simply obtain $((y + x)^m - y^m)\, h_{n-1}$ and the second part of the theorem follows. $\square$

Strong Product

Theorem 5.48. *Let $G = (V, E)$ be a graph. Then the trivariate domination polynomial of the strong product of the complete graph K_n ($n \geq 2$) and the graph G can be obtained by*

$$\mathrm{Y}(G \boxtimes K_n; x, y, z) = \mathrm{Y}\left(G; (y + x)^n - y^n, y^n, 1 + \frac{nx(z - 1)/y}{(1 + x/y)^n - 1}\right).$$

Proof. Let $W \subseteq V(G)$, with $i = |W|$, $i \geq 2$, and $k = \mathrm{iso}(G[W])$, be a set of dominating vertices in G and $y_{ijk} = [x^i y^j z^k]\, \mathrm{Y}(G; x, y, z)$. Then in the product graph every dominating vertex can be replaced by one or more of the n vertices in the same row and the remaining vertices in this row will be dominated. If this vertex is not an isolated vertex in $G[W]$, then this will be counted by

$$((1 + x/y)^n - 1)^{i-k} y^{n(i-k)}.$$

If a vertex in G is dominating and it is isolated in $G[W]$ and we choose exactly one vertex in this row, then this vertex will also be an isolated vertex in $(G \boxtimes K_n)[W]$. If we choose more than one vertex, then these vertices are pairwise adjacent and we obtain no isolated vertex. This will be counted by

$$((1 + x/y)^n - 1 - nx/y + nxz/y)^k\, y^{kn}.$$

Adding these cases together yields:

$$
\begin{aligned}
\mathrm{Y}(G \boxtimes K_n; x, y, z) &= \sum_{i,j,k} y_{ijk}((1 + x/y)^n - 1)^{i-k} y^{n(i+j)}((1 + x/y)^n - 1 - nx/y + nxz/y)^k \\
&= \sum_{i,j,k} y_{ijk}(y^n(1 + x/y)^n - y^n)^i y^{jn}\left(\frac{(1 + x/y)^n - 1 + nx(z - 1)/y}{(1 + x/y)^n - 1}\right)^k \\
&= \sum_{i,j,k} y_{ijk}((y + x)^n - y^n)^i y^{jn}\left(1 + \frac{nx(z - 1)/y}{(1 + x/y)^n - 1}\right)^k \\
&= \mathrm{Y}\left(G; (y + x)^n - y^n, y^n, 1 + \frac{nx(z - 1)/y}{(1 + x/y)^n - 1}\right).
\end{aligned}
$$

$\square$

5.5.3 Special Graph Classes

In this section we investigate the trivariate domination polynomial of some special graph classes. In an edgeless graph every vertex is either an isolated dominating vertex or non-dominated. Therefore, the polynomial is given by

$$Y(E_n; x, y, z) = (1 + xz)^n.$$

The next result for the complete graph is easily verified, too.

Theorem 5.49. *The trivariate domination polynomial of the complete graph satisfies*

$$Y(K_n; x, y, z) = (x + y)^n - y^n + n(z - 1)xy^{n-1} + 1.$$

Proof. If W has at least the size two then no isolated vertex exists in $K_n[W]$, which yields

$$\sum_{k=2}^{n} \binom{n}{k} x^k y^{n-k}.$$

Vertex subsets W with $|W| = 1$ give the term $n(z - 1)xy^{n-1}$ and the theorem follows. $\square$

Theorem 5.50. *Let $K_{n_1,n_2} = (V_1 \cup V_2, E)$ be a complete bipartite graph, with $n_1 = |V_1|$ and $n_2 = |V_2|$. Then*

$$\begin{aligned} Y(K_{n_1,n_2}; x, y, z) = {} & \left((x + y)^{n_1} - y^{n_1}\right)\left((x + y)^{n_2} - y^{n_2}\right) \\ & + y^{n_1}\left((1 + xz)^{n_2} - 1\right) \\ & + y^{n_2}\left((1 + xz)^{n_1} - 1\right) + 1. \end{aligned}$$

Proof. If in both of the two sets V_1 and V_2 at least one vertex is dominating, then all non-dominating vertices are dominated. This yields the first part of the theorem. If only one of the two sets contains dominating vertices, then such a dominating vertex dominates all vertices in the other set and it is an isolated dominating vertex. This yields the theorem. $\square$

Theorem 5.51. *Let $G = (V, E)$ be an $(n, k) - star$. Then*

$$\begin{aligned} Y(S_{n,k}; x, y, z) = {} & y^{n-k}\left((y + x)^k - y^k + k(z - 1)xy^{k-1}\right) \\ & + \left((y + x)^k - y^k\right)\left((x + y)^{n-k} - y^{n-k}\right) + y^k\left((1 + xz)^{n-k} - 1\right) + 1. \end{aligned}$$

Proof. If at least one of the center vertices is dominating, then it dominates all other vertices and if exactly one vertex is dominating, then it is an isolated vertex in $G[W]$. In this case we obtain

$$y^{n-k}\left((y + x)^k - y^k + k(z - 1)xy^{k-1}\right). \tag{5.11}$$

If some of the vertices in the clique are dominating and at least one other vertex is dominating, then the rest of the vertices will be dominated and there will be no isolated vertex in $G[W]$. This leads to

$$\left((y + x)^k - y^k\right)\left((x + y)^{n-k} - y^{n-k}\right). \tag{5.12}$$

Now let no vertex in the center-clique be dominating. All the vertices in the center-clique are dominated by every other vertex which is dominating. But every such dominating vertex is an isolated vertex in $G[W]$. This leads to

$$y^k\left((1 + xz)^{n-k} - 1\right). \tag{5.13}$$

Adding the terms (5.11), (5.12) and (5.13) yields the theorem. $\square$

Corollary 5.52. *Let* $G = (V, E)$ *be an* $(n, 1) - star.$ *Then*

$$\mathrm{Y}(S_{n,1}; x, y, z) = x\,(x + y)^{n-1} + (z - 1)\,xy^{n-1} + y\left((1 + xz)^{n-1} - 1\right) + 1.$$

Lemma 5.53. *Let* P_n *be a path with* n *vertices* $(n \geq 2)$, p_n *be the trivariate domination polynomial of it and* p'_n *be the trivariate domination polynomial under the condition that the first vertex of the path is already dominated. Then*

$$p'_n = x^n + x^{n-1}y + y\sum_{i=1}^{n-1} x^{i-1}p_{n-i}, \tag{5.14}$$

with the initial condition

$$p'_1 = x + y.$$

Proof. Suppose that the first vertex of the path is already dominated. If the first vertex is also dominating, then the second vertex will be dominated. If the first vertex is non-dominating, then we only need to calculate the trivariate domination polynomial in the path P_{n-1} and multiply it with y. The sum of the two cases yields

$$p'_n = xp'_{n-1} + yp_{n-1}.$$

Recursive insertion of this equation yields

$$\begin{aligned}
p'_n &= x(xp'_{n-2} + yp_{n-2}) + yp_{n-1} \\
&= \ldots \\
&= x^n + x^{n-1}yp_0 + \cdots + yp_{n-1} \\
&= x^n + x^{n-1}y + y\sum_{i=1}^{n-1} x^{i-1}p_{n-i}.
\end{aligned}$$

$\square$

Using the last lemma it is possible to prove the next theorem.

Theorem 5.54. *Let* P_n *be a path with* n *vertices,* p_n *be the corresponding trivariate domination polynomial and* $n \geq 3$. *Then*

$$p_n = x^2 p'_{n-2} + xyz(1 + p_{n-2}) + x^2 y\sum_{i=0}^{n-3} p'_i + xy^2 z\sum_{i=0}^{n-3} p_i + 1,$$

where p'_n *is the trivariate domination polynomial of the path* P_n *if the first vertex is already dominated. The initial conditions are*

$$\begin{aligned}
p_0 &= 1 \\
p_1 &= xz + 1 \text{ and} \\
p_2 &= x^2 + 2xyz + 1.
\end{aligned}$$

Proof. Suppose that the first vertex in the path is dominating, then the second vertex can either be dominating or not. If the second vertex is dominating, then it dominates the third

vertex. This case will be counted by $x^2 p'_{n-2}$. If the second vertex is non-dominating, then the first vertex is an isolated vertex in $G[W]$. This case will be counted by $xyzp_{n-2}$.

Suppose now that the i-th vertex is non-dominating. If the vertex $i+1$ is dominating, then it dominates the vertex i. The vertex $i+2$ can either be dominating or not. If it is dominating, then it dominates the next vertex and if it is non-dominating, then the vertex $i+1$ is an isolated vertex in $G[W]$. This case will be counted by $xy(xp'_{n-i} + yzp_{n-i})$, $i \in \{3, \ldots, n\}$.

If the last vertex is dominating, then it dominates the second to the last vertex. This will be counted by xyz.

Taking these different cases together and rearranging the sums yields the theorem.

$$p_n = x^2 p'_{n-2} + xyzp_{n-2} + \sum_{i=3}^{n} \left(xy(xp'_{n-i} + yzp_{n-i}) \right) + xyz + 1$$

$$= x^2 p'_{n-2} + xyz(1 + p_{n-2}) + x^2 y \sum_{i=0}^{n-3} p'_i + xy^2 z \sum_{i=0}^{n-3} p_i + 1.$$

$\square$

Corollary 5.55. *Let P_n be a path with n vertices ($n \geq 2$). Then*

$$p_n = x^n + xy \sum_{i=0}^{n-3} \left(x^{i+1} + yzp_i + x^{i+1}p_{n-i-3} \right)$$

$$+ xyz(1 + p_{n-2}) + xy^2 \sum_{i=1}^{n-3} \sum_{j=1}^{i} x^j p_{i-j} + 1.$$

Proof. Substituting p'_n in Theorem 5.54 with Equation (5.14) yields

$$p_n = x^2 \left(x^{n-2} + y \sum_{i=1}^{n-2} x^{i-1} p_{n-2-i} \right) + xyz(1 + p_{n-2})$$

$$+ x^2 y \sum_{i=0}^{n-3} \left(x^i + y \sum_{j=1}^{i} x^{j-1} p_{i-j} \right) + xy^2 z \sum_{i=0}^{n-3} p_i + 1$$

$$= x^n + x^2 y \sum_{i=1}^{n-2} x^{i-1} p_{n-2-i} + xyz(1 + p_{n-2}) + x^2 y \sum_{i=0}^{n-3} x^i$$

$$+ x^2 y^2 \sum_{i=1}^{n-3} \sum_{j=1}^{i} x^{j-1} p_{i-j} + xy^2 z \sum_{i=0}^{n-3} p_i + 1$$

$$= x^n + xy \sum_{i=0}^{n-3} \left(x^{i+1} + yzp_i + x^{i+1}p_{n-i-3} \right)$$

$$+ xyz(1 + p_{n-2}) + xy^2 \sum_{i=1}^{n-3} \sum_{j=1}^{i} x^j p_{i-j} + 1.$$

$\square$

5.5.4 Y-Unique and Y-Equivalent Graphs

The great variety of graph invariants encoded in the trivariate domination polynomial naturally leads to the question how well this polynomial distinguishes non-isomorphic graphs.

Definition 5.56. *Two graphs G and H are* Y-unique *if* $\mathrm{Y}(G; x, y, z) = \mathrm{Y}(H; x, y, z)$ *implies that H is isomorphic to G.*

The following graphs are Y-unique:

- Paths,

- cycles,

- complete graphs,

- stars,

- star-shaped trees (see Theorem 5.59),

- all trees with up to 18 vertices (shown by computer search),

- all graphs with up to 5 vertices (shown by computer search).

The Y-uniqueness of paths, cycles, complete graphs and stars follows directly from the fact that the number of the components and the degree sequence are encoded in the trivariate domination polynomial (see Theorem 5.41 and 5.42). Figure (5.6) shows the smallest pair of two non-isomorphic graphs with the same trivariate domination polynomial.
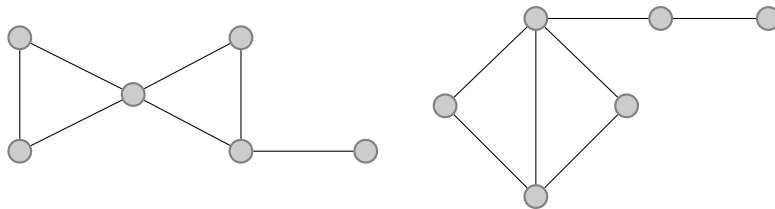
Fig. 5.6: Smallest pair of non-isomorphic graphs with the same trivariate domination polynomial.

Wang and Xu proved in [WX06] that some special trees, the T-shaped trees, are determined by their Laplacian spectrum. We are able to prove similar results for the trivariate domination polynomial. First we define the T- and star-shaped trees.

Definition 5.57. *A* star-shaped tree *is a tree with exactly one vertex of degree m, with $m \geq 3$, and all other vertices have degree one or two. A star-shaped tree is called T-shaped if the vertex v has the degree three.*

Let $m \geq 3$, $2 \leq k_1 \leq \cdots \leq k_m$ and $P_{k_1}, \ldots, P_{k_m}$ be paths with $k_1, \ldots, k_m$ vertices, respectively. Furthermore, let $\bigcap_{i=1}^{m} V(P_{k_i}) = \{v\}$, with pairwise disjunct vertex sets $V(P_{k_1}) \backslash \{v\}, \ldots, V(P_{k_m}) \backslash \{v\}$. Then $T(k_1, \ldots, k_m) = P_{k_1} \cup \cdots \cup P_{k_m}$ is a star-shaped tree.

Theorem 5.58. *Let G and H be two T-shaped trees with n vertices of the form $T(l_1, l_2, l_3)$ and $T(k_1, k_2, k_3)$, respectively. Then*

$$G \cong H \Leftrightarrow \mathrm{Y}(G; x, y, z) = \mathrm{Y}(H; x, y, z).$$
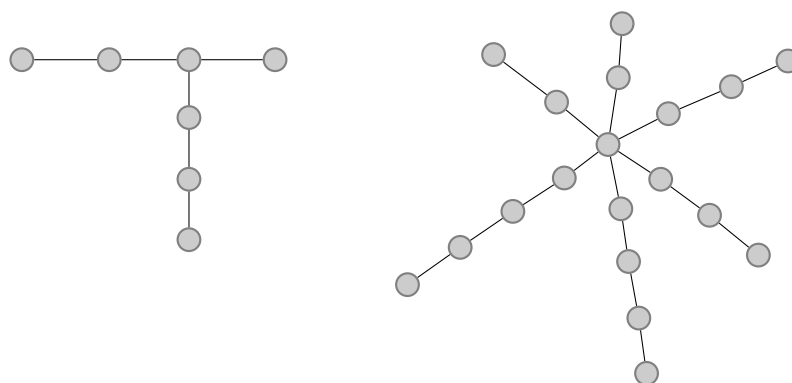
Fig. 5.7: A T- and a star-shaped tree.

Proof. $\Rightarrow$: This direction follows directly from the definition of $\mathrm{Y}(G; x, y, z)$.

$\Leftarrow$: To prove this direction we show that if we have a trivariate domination polynomial of a T-shaped tree, then the construction of the corresponding graph is unique. Every T-shaped tree has the degree sequence $(1, 1, 1, 2, \ldots, 2, 3)$. This sequence can easily be extracted from the trivariate domination polynomial (see Theorem 5.42). Two T-shaped trees of a given degree sequence can be distinguished by the length of their branches.

Let $n_l(G)$ be the number of branches of length l in G. Then $n_1(G) = 3 - [x^2 y^1 z^0] \mathrm{Y}(G; x, y, z)$, because the only way to choose two adjacent vertices that dominate exactly one vertex, is to choose a leaf and its neighbor, if the neighbor is not the center vertex v. Therefore, $[x^2 y^1 z^0] \mathrm{Y}(G; x, y, z)$ gives the number of branches which lengths are equal or greater than two.

The number of branches of length l is:

$$n_l(G) = 3 - \sum_{i=1}^{l-1} n_i(G) - [x^{l+1} y^1 z^0] \mathrm{Y}(G; x, y, z). \qquad (5.15)$$

The construction of the graph is unique and therefore the theorem is proved. $\qquad \square$

We can use the technique of the proof of the last theorem to prove a generalization of it.

Theorem 5.59. *Let G and H be two star-shaped trees with n vertices of the form $T(l_1, \ldots, l_m)$ and $T(k_1, \ldots, k_m)$, respectively. Then*

$$G \cong H \Leftrightarrow \mathrm{Y}(G; x, y, z) = \mathrm{Y}(H; x, y, z).$$

Proof. The argumentation of this proof is the same as in the previous theorem. $\qquad \square$

For general trees, we have the following conjecture.

Conjecture 5.60. *Trees are determined by their trivariate domination polynomial.*

6 The Connected Domination Polynomial

In Chapter 5 we demand that the dominating vertices must be dominated by another dominating vertex. This means that the induced subgraph of the dominating vertices $G[W]$ has no isolated vertices, but the graph $G[W]$ can have more than one component. If we now ask for dominating sets such that the dominating set has to induce a connected subgraph, then we obtain the so-called connected dominating sets. The concept of connected dominating sets was first introduced by Sampathkumar and Walikar [SW79]. They have applications in wireless sensor networks, wireless ad hoc networks and in the connection with some broadcast problems (see [GK98]).

Definition 6.1. *Let $G = (V, E)$ be a graph. Then the vertex subset $W \subseteq V$ is a connected dominating set if $N[W] = V$ and $k(G[W]) = 1$.*

The first problem we are interested in is finding the size of a smallest connected dominating set of a given graph.

Definition 6.2. *[SW79] The connected domination number is the size of a smallest connected dominating set of the graph G and is denoted by $\gamma_c(G)$.*

The connected domination number has a connection to the maximum number of leaves in a spanning tree of G. It was first observed by Hedetniemi and Laskar [HL84].

Definition 6.3. *Let $G = (V, E)$ be a graph. Then the* maximum leaf number $l(G)$ *is the largest possible number of leaves in a spanning tree of G.*

Theorem 6.4. *[HL84] Let $G = (V, E)$ be a graph and $l(G)$ the maximum leaf number of G. Then the following equation holds:*

$$|V| = \gamma_c(G) + l(G).$$

The problem to decide if a graph has a connected dominating set of size at most k is a NP-complete problem [GJ79]. The corresponding counting problem is in $\sharp$P. For the computation of the connected domination number, a wide range of approximation algorithms, lower and upper bounds are known (see [GK98]). The fastest algorithm known so far to compute the connected domination number has a running time of $O(1.9407^n)$ [FGK08]. But we are interested in counting all connected dominating sets of the graph. On this account, we define the connected domination polynomial which counts the number of the connected dominating sets of different sizes.

Definition 6.5. *Let $G = (V, E)$ be a graph and $d_k^c(G)$ be the number of the connected dominating sets of size k of the graph G. Then the connected domination polynomial is defined as*

$$D_c(G, x) = \sum_{k=1}^{n} d_k^c(G) x^k.$$

Theorem 6.6. *Computing the connected domination polynomial of a graph is NP-hard.*

Proof. This follows immediately from a result of Garey and Johnson [GJ79].   □

Remark 6.7. *The connected domination polynomial can also be written in the following way:*

$$D_c(G, x) = \sum_{\substack{W \subseteq V \\ G[W]-connected \\ N[W]=V}} x^{|W|}.$$

Remark 6.8. *Every connected dominating set is a dominating set, but not every dominating set is connected. Furthermore, every connected dominating set of size greater or equal to two is a total dominating set, but not vice versa. So $d_k^c(G) \leq d_k^t(G) \leq d_k(G)$ holds, for $k \in \{2, 3, \ldots, n\}$.*

Remark 6.9. *Let $G = (V, E)$ be a graph and $k(G) \neq 1$. Then*

$$D_c(G, x) = 0.$$

Theorem 6.10. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two vertex-disjoint graphs. Then*

$$D_c(G * H, x) = \left((1+x)^{|V(G)|} - 1\right)\left((1+x)^{|V(H)|} - 1\right) + D_c(G, x) + D_c(H, x).$$

Proof. Every vertex in G is adjacent to every vertex in H. Hence, if at least one vertex in G and at least one vertex in H are dominating, then these vertices are a connected dominating set in $G * H$. If no vertex in G is dominating, then every connected dominating set in H dominates the whole graph and vice versa.   □

As a direct consequence of the previous theorem, we obtain the following four corollaries which show the calculation in special graph classes.

Corollary 6.11. *Let $G = (V, E)$ be the (n, k)-star $S_{n,k}$. Then*

$$D_c(S_{n,k}, x) = \left((1+x)^k - 1\right)(1+x)^{n-k}.$$

Corollary 6.12. *Let $G = (V, E)$ be the star S_n, with $n \geq 3$. Then*

$$D_c(S_n, x) = x(1+x)^{n-1}.$$

Corollary 6.13. *Let $G = (V, E)$ be the wheel graph W_n, with $n \geq 4$. Then*

$$D_c(W_n, x) = D_c(C_{n-1}, x) + x(1+x)^{n-1}.$$

Corollary 6.14. *Let $G = (V, E)$ be the fan graph F_n, with $n \geq 3$. Then*

$$D_c(F_n, x) = D_c(P_{n-1}, x) + x(1+x)^{n-1}.$$

Theorem 6.15. *Let $G = (V(G), E(G))$ be a connected graph and $H = (V(H), E(H))$ be a graph which is vertex-disjoint to G. Furthermore, let $|V(G)| = n_G$ and $|V(H)| = n_H$. Then*

$$D_c(G \circ H, x) = x^{n_G}(1+x)^{n_G n_H}.$$

Proof. Every vertex of G is an articulation in the corona graph and therefore every connected dominating set has to contain all vertices of G. All vertices of the n_G copies of H are adjacent to such a dominating vertex and therefore they can either be in a connected dominating set or not. $\qquad\square$

The next theorem shows a result for the sum over all connected domination polynomials of the vertex induced subgraphs of a given graph G.

Theorem 6.16. *Let $G = (V, E)$ be a connected graph. Then*

$$\sum_{W \subseteq V} (-1)^{|W|} \, \mathrm{D}_c(G[W], x) = 1 + (-x)^{|V|}.$$

Proof. See [KPT13] and the proof of Theorem 5.5. $\qquad\square$

Together with the type λ_G (see Definition 2.8) of the graph G we obtain the next corollary.

Corollary 6.17. *Let $G = (V, E)$ be a graph. Then*

$$\sum_{W \subseteq V} (-1)^{|W|} \, \mathrm{D}_c(G[W], x) = 1 + \sum_{i \in \lambda_G} (-x)^i. \qquad (6.1)$$

Proof. Let $V_1, V_2, \ldots, V_k$ be the vertex sets of the k components of the graph G. Then we can write the left hand side of the corollary in the following way:

$$\sum_{W \subseteq V} (-1)^{|W|} \, \mathrm{D}_c(G[W], x) = 1 + \sum_{i=1}^{k} \sum_{\substack{W \subseteq V_i \\ W \neq \emptyset}} (-1)^{|W|} \, \mathrm{D}_c(G[W], x)$$

$$= 1 + \sum_{i=1}^{k} (-x)^{|V_i|}.$$

$\qquad\square$

Applying the Möbius inversion to Equation (6.1) yields the following corollary.

Corollary 6.18. *Let $G = (V, E)$ be a graph. Then*

$$\mathrm{D}_c(G, x) = \sum_{W \subseteq V} (-1)^{|W|} \sum_{i \in \lambda_{G[W]}} (-x)^i.$$

A consequence of the previous corollary is that we can calculate the number of connected dominating sets of a graph as the sum over the difference between the number of the even and the odd components of the vertex induced subgraphs of G.

Corollary 6.19. *Let $G = (V, E)$ be a graph, $k_e(G)$ the number of the components of even order and $k_o(G)$ the number of components of odd order. Then the number of connected dominating sets in G can be calculated with*

$$\mathrm{D}_c(G, 1) = \sum_{W \subseteq V} (-1)^{|W|} \left(k_e(G[W]) - k_o(G[W]) \right).$$

6.1 Recurrence Equations and Separating Vertex Sets

With the Corollary 6.18, we easily obtain a recurrence equation for the connected domination polynomial.

Theorem 6.20. *Let $G = (V, E)$ be a connected graph and u be a vertex of the graph. Then*

$$\mathrm{D}_c(G, x) = \mathrm{D}_c(G - u, x) + \sum_{\substack{u \in W \subseteq V \\ G[W]-conn.}} \left((-1)^{|W|} \mathrm{D}_c(G - N[W], x) + 0^{|V|-|N[W]|} x^{|W|} \right).$$

Proof. Using Corollary 6.18, we obtain:

$$\mathrm{D}_c(G, x) = \sum_{W \subseteq V} (-1)^{|W|} \sum_{i \in \lambda_{G[W]}} (-x)^i$$

$$= \sum_{W \subseteq V \setminus \{u\}} (-1)^{|W|} \sum_{i \in \lambda_{G[W]}} (-x)^i + \sum_{u \in W \subseteq V} (-1)^{|W|} \sum_{i \in \lambda_{G[W]}} (-x)^i$$

$$= \mathrm{D}_c(G - u, x) + \sum_{\substack{u \in W \subseteq V \\ G[W]-conn.}} (-1)^{|W|} \sum_{U \subseteq V \setminus N[W]} (-1)^{|U|} \sum_{i \in \lambda_{G[W \cup U]}} (-x)^i$$

$$= \mathrm{D}_c(G - u, x) + \sum_{\substack{u \in W \subseteq V \\ G[W]-conn.}} (-1)^{|W|} \sum_{U \subseteq V \setminus N[W]} (-1)^{|U|} \left((-x)^{|W|} + \sum_{i \in \lambda_{G[U]}} (-x)^i \right)$$

$$= \mathrm{D}_c(G - u, x) + \sum_{\substack{u \in W \subseteq V \\ G[W]-conn.}} (-1)^{|W|} \sum_{U \subseteq V \setminus N[W]} (-1)^{|U|} (-x)^{|W|}$$

$$+ \sum_{\substack{u \in W \subseteq V \\ G[W]-conn.}} (-1)^{|W|} \sum_{U \subseteq V \setminus N[W]} (-1)^{|U|} \sum_{i \in \lambda_{G[U]}} (-x)^i$$

$$= \mathrm{D}_c(G - u, x) + \sum_{\substack{u \in W \subseteq V \\ G[W]-conn. \\ N[W]=V}} x^{|W|} + \sum_{\substack{u \in W \subseteq V \\ G[W]-conn.}} (-1)^{|W|} \mathrm{D}_c(G - N[W], x)$$

$$= \mathrm{D}_c(G - u, x) + \sum_{\substack{u \in W \subseteq V \\ G[W]-conn.}} \left((-1)^{|W|} \mathrm{D}_c(G - N[W], x) + 0^{|V|-|N[W]|} x^{|W|} \right).$$

$\square$

Another way to obtain a recurrence equation is to look at a specific vertex and its neighborhood. For such a sort of recurrence equation we need the definition of $p_u^c(G)$.

Definition 6.21. *Let $G = (V, E)$ be a graph. Then $p_u^c(G)$ is the connected domination polynomial of the graph $G - N[u]$ under the condition that all vertices in $N(u)$ are dominated by a vertex in $G - N[u]$.*

With this definition, we can prove the following theorem.

Theorem 6.22. *Let $G = (V, E)$ be a graph and u be a vertex of the graph. Then*

$$\mathrm{D}_c(G, x) = \mathrm{D}_c(G - u, x) + x \mathrm{D}_c(G/u, x) - (1 + x) p_u^c(G) + x^2 \sum_{w \in N(u)} 0^{|V|-|N[\{u,w\}]|}.$$

Proof. The argumentation is the same as in the proof of Theorem 5.14, except for the sum. Suppose that u and $v \in N(u)$ are dominating vertices and no other vertex in $N[\{u,v\}]$ is dominating. Then $\{u,v\}$ is only a connected dominating set if and only if $N[\{u,v\}] = V$. $\quad\square$

The "problem" with this recurrence equation is that the polynomial $p_u^c(G)$ is a new polynomial. But in some special cases this polynomial becomes zero and we obtain some simple recurrence equations.

Corollary 6.23. *Let $G = (V, E)$ be a graph. If two vertices $u, v \in V$ exist with $N(v) \subseteq N(u)$, then*

$$\mathrm{D}_c(G, x) = \mathrm{D}_c(G - u, x) + x\,\mathrm{D}_c(G/u, x) + x^2 \sum_{w \in N(v)} 0^{|V| - |N[\{u,w\}]|}.$$

Corollary 6.24. *Let $G = (V, E)$ be a graph. If two vertices $u, v \in V$ exist with $N[v] \subseteq N[u]$ and $N(u)$ forms a clique in the graph, then*

$$\mathrm{D}_c(G, x) = (1 + x)\,\mathrm{D}_c(G - u, x) + x^2 \sum_{w \in N(v)} 0^{|V| - |N[\{u,w\}]|}.$$

It is also possible to prove a recurrence equation with respect to the deletion of a vertex, the deletion of edges between adjacent vertices and the combination of these two operations.

Theorem 6.25. *Let $G = (V, E)$ be a graph. Then*

$$\mathrm{D}_c(G, x) = \mathrm{D}_c(G - u, x) + \mathrm{D}_c(G \odot u, x) - \mathrm{D}_c(G \circledcirc u, x).$$

Proof. To prove the theorem, we use the idea of the proof of the Theorem 5.22. Applying the Equations (2.1) to the Theorem 6.22 yields

$$\begin{aligned}
\mathrm{D}_c(G, x) - \mathrm{D}_c(G - u, x) =\ & x\,\mathrm{D}_c(G/u, x) - (1 + x)p_u^c(G) \\
& + x^2 \sum_{w \in N(u)} 0^{|V| - |N[\{u,w\}]|} \\
=\ & x\,\mathrm{D}_c((G \odot u)/u, x) - (1 + x)p_u^c(G \odot u) \\
& + x^2 \sum_{w \in N(u)} 0^{|V| - |N[\{u,w\}]|}.
\end{aligned} \tag{6.2}$$

Now we apply the Theorem 6.22 to the graph $G \odot u$:

$$\begin{aligned}
\mathrm{D}_c(G \odot u, x) - \mathrm{D}_c((G \odot u) - u, x) =\ & x\,\mathrm{D}_c((G \odot u)/u, x) - (1 + x)p_u^c(G \odot u) \\
& + x^2 \sum_{w \in N(u)} 0^{|V| - |N_{G \odot u}[\{u,w\}]|}.
\end{aligned} \tag{6.3}$$

Observe that $N_G(u) = N_{G \odot u}(u)$ and $N_G[\{u,w\}] = N_{G \odot u}[\{u,w\}]$. These two observations together with the Equations (6.2) and (6.3) give the theorem. $\quad\square$

Corollary 6.26. *Let $G = (V, E)$ be a graph such that u and w are adjacent vertices of the graph with $N[w] \subseteq N[u]$. Then*

$$\mathrm{D}_c(G, x) = \mathrm{D}_c(G - u, x) + \mathrm{D}_c(G \odot u, x).$$

Remark 6.27. *Let $G = (V, E)$ be a graph and $e \in E$ be an edge of the graph. Then every connected dominating set in $G - e$ is also connected dominating in G.*

If a graph has a separating vertex set, then it is possible for many graph polynomials to find a splitting formula. In the case of the connected domination polynomial we can prove such a result if the separating vertex set is a clique.

Theorem 6.28. *Let* $G = (V, E)$ *be a graph,* $\{G_1, G_2, X\}$ *be a splitting of* G *and* X *induces a clique in the graph. Then*

$$\mathrm{D}_c(G, x) = \frac{1}{(1+x)^2} \sum_{\substack{Y \subseteq X \\ |Y| \geq 1}} x^{|Y|-2} \, \mathrm{D}_c(G_1 - (X - Y) \rhd\!\!-\!\!\circ\, Y, x)$$

$$\mathrm{D}_c(G_2 - (X - Y) \rhd\!\!-\!\!\circ\, Y, x).$$

Proof. In the separating vertex set X, there must be at least one dominating vertex and therefore the other vertices in the separating set are dominated. So, if in the graph G_1 the vertices in $Y \subseteq X$ are dominating, we can remove the vertices in $X - Y$ from G_1. Additionally, we can contract the vertices of Y to a new vertex y and remove parallel edges because all vertices in X are pairwise adjacent and all vertices in Y must be dominating. Then we add a new adjacent vertex to the vertex y, to guarantee that the vertex y is dominating. The new vertex yields the term $(1 + x)$ in the polynomial because the additional vertices can either be dominating or not, but the vertex y occurs in every connected dominating set in $G_1 - (X - Y) \rhd\!\!-\!\!\circ\, Y$. Then we multiply the polynomial with the corresponding one of G_2 and $x^{|Y|}$ and divide it by $x^2(1 + x)^2$ for the double counted vertices. $\qquad\square$

Corollary 6.29. *Let* $G = (V, E)$ *be a graph and* $\{G_1, G_2, \{v\}\}$ *be a splitting of* G. *Then*

$$\mathrm{D}_c(G, x) = \frac{\mathrm{D}_c(G_1 + \{v, \cdot\}, x) \, \mathrm{D}_c(G_2 + \{v, \cdot\}, x)}{x(1 + x)^2}.$$

6.2 Irrelevant Edges and Vertices

In this section we characterize the essential vertices and the irrelevant edges and vertices of a graph.

Definition 6.30. *Let* $G = (V, E)$ *be a connected graph. A vertex* v *of the graph is called* essential *if* $\mathrm{D}_c(G - v, x) = 0$ *and it is called* irrelevant *if*

$$\mathrm{D}_c(G, x) = (1 + x) \, \mathrm{D}_c(G - v, x).$$

Moreover, an edge $e \in E$ *of the graph is called* irrelevant *if* $\mathrm{D}_c(G, x) = \mathrm{D}_c(G - e, x)$.

An essential vertex is included in every connected dominating set of the graph. This leads to the following lemma.

Lemma 6.31. *A vertex* v *of a connected graph* $G = (V, E)$ *is essential if and only if* v *is an articulation.*

Proof. The graph is disconnected after the removal of the articulation v and therefore $\mathrm{D}_c(G - v, x) = 0$. Thus, every articulation is essential.

Suppose v is an essential vertex, but not an articulation. Then the set $V \backslash v$ is a connected dominating set in G, which is a contradiction to the assumption that v is essential. $\qquad\square$

The characterization of the essential vertices together with Theorem 6.25 yields the next corollary.

Corollary 6.32. *Let $G = (V, E)$ be a graph and u be an articulation. Then*

$$\mathrm{D}_c(G, x) = \mathrm{D}_c(G \odot u, x).$$

The next theorem characterizes the irrelevant edges of a graph. We call two essential vertices *essentially connected* if there exist a path between them which only consists of essential vertices. If not such a path exists, then we call the two vertices *essentially non-connected.*

Theorem 6.33. *Let $G = (V, E)$ be a connected graph and $e = \{u, v\}$ be an edge of the graph. Then e is an irrelevant edge if and only if its end vertices are adjacent to articulations which are essentially connected in $G - e$.*

Proof. "$\Leftarrow$": Suppose that the articulation w is the common neighbor of the vertices u and v. Then u and v will be dominated from w and the connectedness of dominating vertices depends not on the existence of the edge e.

Suppose now that the vertex u is adjacent to the articulation w and v adjacent to the articulation x (see Figure 6.1). Furthermore, let x and w be essentially connected. Then every dominating set which is connected in G is also connected in $G - e$.
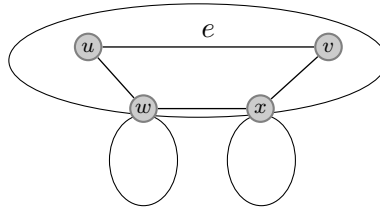
Fig. 6.1: Graph with an irrelevant edge e.

"$\Rightarrow$": To prove this direction we distinguish three different cases: The vertices u and v are not adjacent to an articulation, one of them is adjacent to an articulation and both vertices are adjacent to an articulation but the articulations are essentially non-connected.

1. Suppose that $e = \{u, v\}$ is an irrelevant edge of the graph G and no neighbor of u and no neighbor of v is an articulation. Then it exists at least one path between u and v without e in G, otherwise $G - e$ is not connected. It remains to show that there exist at least one connected dominating set in G, which is not connected dominating in $G - e$. Let $W \subseteq V$ be a connected dominating vertex set of G, such that $u, v \in W$, but the vertices u and v are not connected by a path of dominating vertices. Such a connected dominating vertex set exists because of the fact that u and v are essentially non-connected in $G - e$. Then W is a connected dominating set in G, but not in $G - e$ and therefore e is not irrelevant.

2. To show this case we can use the argumentation of the case one (see Figure 6.2).

3. Suppose that $e = \{u, v\}$ is an irrelevant edge of the graph G and they are adjacent to two articulations w and x which are essentially non-connected (see Figure 6.2). Then there exist at least one non-essential vertex on every path between w and x. If these vertices are non-dominating, then the resulting dominating set is non-connected in $G - e$ and therefore the edge e is not irrelevant.

$\square$

Remark 6.34. *If e and f are two irrelevant edges in G, then f is not necessarily an irrelevant edge in $G - e$ (e.g. see Figure 6.3).*
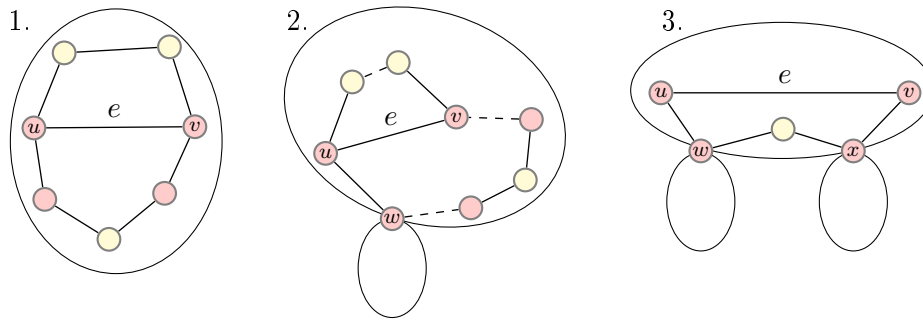
Fig. 6.2: Dominating vertex sets (red) which are connected in G, but non-connected in $G - e$.
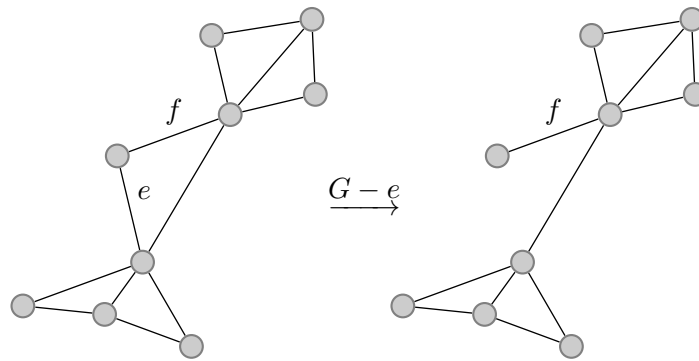
Fig. 6.3: Graph with two irrelevant edges e and f (left), whereas f is not irrelevant in $G - e$.

Lemma 6.35. *Let $G = (V, E)$ be a graph, $v \in V$ be a vertex, $\mathcal{W}_v$ be the set of connected dominating set of $G - v$ and $\mathcal{W}$ be the set of connected dominating sets of G. The vertex v of G is irrelevant if and only if $\mathcal{W} = \mathcal{W}_v \cup \{W \cup \{v\} : W \in \mathcal{W}_v\}$.*

Proof. The proof follows directly from the definition of the irrelevant vertices. □

Theorem 6.36. *Let $G = (V, E)$ be a graph. A vertex $v \in V$ is irrelevant if and only if every incident edge is an irrelevant edge in G and the adjacent articulations induce a connected subgraph.*

Proof. "$\Leftarrow$": Suppose that all incident edges of $v \in V$ are irrelevant. Let $e = \{u, v\} \in E$ be such an edge, then u must be adjacent to an articulation and therefore it will be dominated from this vertex. Because of the fact that all adjacent articulations induce a connected subgraph, the vertex v is irrelevant.

"$\Rightarrow$": Suppose now that $v \in V$ is an irrelevant vertex and $e = \{u, v\} \in E$ is an incident edge which is not irrelevant in G. Then u is not adjacent to an articulation which is essentially connected to an adjacent articulation of v. Therefore, we can construct a connected dominating set W, with $u, v \in W$ and all adjacent articulations of v are in W, such that v is an articulation in $G[W]$ (see Figure 6.4), which is a contradiction (follows from Lemma 6.35).

□
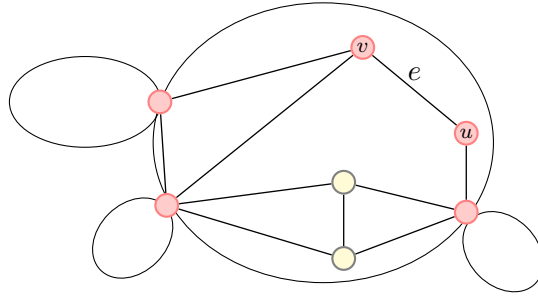
Fig. 6.4: Graph with non-irrelevant vertex v.

6.3 Special Graph Classes

For many graph classes, the calculation of the connected domination polynomial is easy. In this section, formulas and recurrence equations will be proved for several graph classes.

6.3.1 Complete and Nearly Complete Graphs

The class of the complete graphs is the easiest graph class for the most problems. Every non-empty subset of the vertex set is a connected dominating set. This leads directly to

$$\mathrm{D}_c(K_n, x) = (1+x)^n - 1. \tag{6.4}$$

In Section 2.2 the k-bounded complete graphs are introduced (see Definition 2.12). We can generalize the Equation (6.4) for the complete graph to the simple k-bounded complete graphs and obtain the following theorem.

Theorem 6.37. *Let $G = (V, E)$ be a simple k-bounded complete graph K_n^k with n vertices and the type $\Lambda(K_n^k) = [n_0, n_1, n_2, \ldots, n_l]$. Then*

$$\mathrm{D}_c(K_n^k, x) = ((1+x)^{n_0} - 1)(1+x)^{n-n_0} + \sum_{i=1}^{l-1}\left[((1+x)^{n_i} - 1)\left(\prod_{j=i+1}^{l}(1+x)^{n_j} - 1\right)\right].$$

Proof. If at least one vertex of the clique of size n_0 is in the dominating vertex set, then the remaining $n - n_0$ vertices can either be dominating or not. The so constructed dominating set W induces a connected graph. Let now no vertex in $V_0, \ldots, V_{i-1}$ be dominating, but at least one vertex in V_i. Then this vertex dominates all vertices in $V \backslash V_i$, but the rest of the vertices in V_i will not be dominated and the set induces no connected graph. Therefore, at least one vertex in $V_{i+1}, \ldots, V_l$ must be dominating and the theorem follows. $\qquad\square$

The second way to obtain nearly complete graphs is to remove a matching from the complete graph.

Theorem 6.38. *Let $K_n = (V, E)$ be a complete graph and $M \subset E$ be a matching of the graph with $m = |M|$. Then*

$$\mathrm{D}_c(K_n - M, x) = (1+x)^n - mx^2 - 2mx - 1.$$

Proof. Every vertex subset W of size of at least one is a connected dominating set, except for those vertex subsets of size two which consist of two non-adjacent vertices and the vertex subsets which consist of one vertex with degree $n - 2$. There are m possibilities to choose such a vertex pair and $2m$ possibilities to choose a single vertex with degree $n - 2$. $\qquad\square$

Corollary 6.39. *Let $K_n = (V, E)$ be a complete graph and $M \subset E$ be a perfect matching of it. Then*

$$D_c(K_n - M, x) = (1 + x)^n - \frac{n}{2}x^2 - nx - 1.$$

Theorem 6.40. *Let $G = (V, E)$ be a complete graph with k holes. Let n_i be the size of the i-th hole in the graph and $m_j = \sum_{i=1}^{j} n_i$, for $j \in \{1, \ldots, k\}$. Then*

$$D_c(G, x) = (1 + x)^{n - m_k} + \sum_{i=1}^{k} \left[\left((1 + x)^{n - m_i} - 1 \right) \left((1 + x)^{n_i} - 1 \right) + D_c(C_{n_i}, x) \right] - 1.$$

Proof. Apply Theorem 6.10 iteratively to the holes and the rest of the graph. □

Corollary 6.41. *Let $G = (V, E)$ be a complete graph with k anti-holes. Let n_i be the size of the i-th anti-hole in the graph and $m_j = \sum_{i=1}^{j} n_i$, for $j \in \{1, \ldots, k\}$. Then*

$$D_c(G, x) = (1 + x)^{n - m_k} + \sum_{i=1}^{k} \left[\left((1 + x)^{n - m_i} - 1 \right) \left((1 + x)^{n_i} - 1 \right) + D_c(\overline{C_{n_i}}, x) \right] - 1.$$

6.3.2 Complete and Nearly Complete Bipartite Graphs

Theorem 6.42. *Let $K_{m,n} = (V_1 \cup V_2, E)$ be a complete bipartite graph with $|V_1| = n$ and $|V_2| = m$. Then*

$$D_c(K_{m,n}, x) = ((1 + x)^n - 1) ((1 + x)^m - 1).$$

Proof. If we choose at least one vertex from V_1 and V_2, then all vertices are dominated and the vertex subset is connected. □

Theorem 6.43. *Let $G = (V_1 \cup V_2, E)$ be a 1-bounded bipartite graph with $|V_1| = n$, $|V_2| = m$ and let k be the number of vertices in V_1 with degree $m - 1$. Then*

$$\begin{aligned}
D_c(G, x) = {} & \left((1 + x)^{n-k} - 1 \right) \left((1 + x)^{m-k} - 1 \right) (1 + x)^{2k} \\
& + ((1 + x)^k - 1 - kx)(1 + x)^k \left[\left((1 + x)^{n-k} - 1 \right) + \left((1 + x)^{m-k} - 1 \right) \right] \\
& + \left((1 + x)^k - 1 - kx \right)^2 - \binom{k}{2} x^4.
\end{aligned}$$

Proof. To prove the theorem, we distinguish three cases with respect to the number of the dominating vertices of degree m in V_1 and of degree n in V_2. Let $W_1 \subseteq V_1$ be the vertices in V_1 which have degree m and $W_2 \subseteq V_2$ be the vertices in V_2 which have degree n. Additionally let $U_i = V_i \backslash W_i$, for $i \in \{1, 2\}$. If in W_1 and as well in W_2 at least one vertex is dominating, then all other vertices are dominated and the dominating vertices are connected. If only in W_1 at least one vertex is dominating but no vertex in W_2 is dominating, then in U_2 at least two vertices must be dominating to dominate all vertices in V_1. This case will be counted by

$$((1 + x)^k - 1 - kx) \left((1 + x)^{n-k} - 1 \right) (1 + x)^k.$$

The same argumentation holds if only in W_2 at least one vertex is dominating, but no vertex in W_1 is dominating. If in W_1 and in W_2 no vertex is dominating, then in U_1 and U_2 at least two vertices must be dominating to dominate all other vertices and to induce a connected

vertex subset. But if we choose two vertices in U_1 and the two corresponding vertices in U_2, then the induced subgraph is the graph $P_2 \cup P_2$ which is not connected. Therefore, the polynomial for this case is

$$\left((1+x)^k - 1 - kx\right)^2 - \binom{k}{2}x^4$$

and the theorem follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 6.44. *Let $K_{n,n} = (V_1 \cup V_2, E)$ be a complete bipartite graph, with $|V_1| = |V_2| = n$ and $M \subset E$ be a perfect matching of the graph. Then*

$$D_c(K_{n,n} - M, x) = ((1+x)^n - 1 - nx)^2 - \binom{n}{2}x^4.$$

6.3.3 Trees, Paths and Cycles

Theorem 6.45. *Let $G = (V, E)$ be a tree with n vertices and k leaves. Then the connected domination polynomial can be calculated by*

$$D_c(G, x) = x^{n-k}(1+x)^k.$$

Proof. Let L be the set of the leaves of the tree. Then $V \backslash L$ is a subset of every connected dominating set because if one vertex from $V \backslash L$ is non-dominating, then the dominating set cannot be connected. The leaves of the tree can either be dominating or not. $\qquad\square$

A nice consequence of the previous theorem is that we obtain a simple formula for the path P_n.

Corollary 6.46. *Let $G = (V, E)$ be the path P_n, then*

$$D_c(P_n, x) = x^n + 2x^{n-1} + x^{n-2}.$$

A more general graph class are the simple k-paths (see Definition 2.24).

Definition 6.47. *Let $G = (V, E)$ be a simple k-path. Then $f_n^{(k)}$ is the connected domination polynomial of the simple k-path $P_n^{(k)}$ under the condition that the first k vertices are already dominated.*

Theorem 6.48. *Let $G = (V, E)$ be a simple k-path $P_n^{(k)}$ with $n \geq k + 2$. Then*

$$D_c(P_n^{(k)}, x) = x \sum_{i=1}^{k+1} f_{n-i}^{(k)}$$

and

$$f_n^{(k)} = x \sum_{i=1}^{k} f_{n-i}^{(k)}.$$

The initial conditions are

$$f_i^{(k)} = (1+x)^i, \quad \forall i \in \{1, \ldots, k\}.$$

Proof. At least one of the first $k+1$ vertices in the k-path must be dominating and if a vertex is dominating, then the next k vertices will be dominated. So we obtain the first recurrence equation.

If the first k vertices are already dominated, then at least one of these k vertices must be dominating to obtain a connected dominating set. This leads to the second recurrence equation. If the k-path has at most k vertices which are already dominated, then they can either be dominating or not. $\square$

Remark 6.49. *It is also possible to use Theorem 6.28 to calculate the connected domination polynomial of a k-path $P_n^{(k)}$.*

Remark 6.50. *Let $\mathrm{D}_c(P_n^{(k)}, x)$ be the connected domination polynomial of the simple k-path $P_n^{(k)}$. Then $\mathrm{D}_c(P_n^{(k)}, 1)$ yields the number of 01-words of length n, with no subword with k consecutive zeros, except of the first and the last k digits.*

For the connected domination polynomial it is also possible to prove a short equation for the cycle in contrast to other graph polynomials like the domination polynomial.

Theorem 6.51. *Let $G = (V, E)$ be the cycle C_n. Then*

$$\mathrm{D}_c(C_n, x) = x^n + nx^{n-1} + nx^{n-2}.$$

Proof. If one vertex of the cycle is non-dominating, then one of both neighbors must be dominating and there cannot be two non-adjacent non-dominating vertices. The only possible way to choose non-dominating vertices is to choose one single vertex or two adjacent vertices. With these considerations the theorem follows. $\square$

Theorem 6.52. *Let $G = (V, E)$ be the anti-cycle $\overline{C_n}$. Then*

$$\mathrm{D}_c(G, x) = (1 + x)^n - 1 - nx - 2nx^2 - nx^3.$$

Proof. Every vertex of the anti-cycle is non-adjacent to two other vertices. Therefore every vertex subset of size one is a non-dominating set, and every vertex subset of size two with two non-adjacent vertices is not connected. Additionally, every vertex subset of size two, where the two vertices have a common non-adjacent vertex, is non-dominating. Furthermore, if we add the common non-neighbor to these vertex subsets we obtain a dominating set which is not connected. All other vertex subsets are connected dominating sets. $\square$

6.3.4 Some Product Graphs

In this section some equations for the calculation of the connected domination polynomial in product graphs will be given. The first product of interest is the strong product. First we prove a theorem about the strong product of a complete graph and an arbitrary graph.

Theorem 6.53. *Let $G = (V, E)$ be a graph. Then the connected domination polynomial of the strong product of the complete graph K_n and the graph G can be obtained by*

$$\mathrm{D}_c(G \boxtimes K_n, x) = \mathrm{D}_c\left(G, (1 + x)^n - 1\right).$$

Proof. Let $W \subseteq V(G)$ be a connected dominating set in G. Then vertex set $\{(v, u) : v \in W, u \in V(K_n)\}$ is connected dominating in the product graph $G \boxtimes K_n$. This yields the theorem. $\square$

It is already known that the connected domination number γ_c of the ladder graph $(P_n \square P_2)$ equals n [Weid]. There are also some known recurrence equations for graph polynomials, e.g. for the chromatic polynomial, the independence polynomial and the matching polynomial. The next theorem yields a result for the connected domination polynomial for ladder graphs.

Theorem 6.54. *Let p_n be the connected domination polynomial of the product graph $P_n \square P_2$ ($n \geq 3$), p'_n be the polynomial under the condition that one vertex in the last row is dominating and p''_n be the polynomial under the condition that all vertices in the last row of the product graph are dominating. Then*

$$p_n = x(1+x)p'_{n-1} + (1+x)^2 p''_{n-1},$$
$$p'_n = x p'_{n-1} + 2x p''_{n-1},$$
$$p''_n = x^2(p'_{n-1} + p''_{n-1}).$$

The initial conditions are

$$p'_2 = 2x^2 + 2x^3 \text{ and } p''_2 = x^2 + 2x^3 + x^4.$$

Proof. At least one vertex of the second to last row is included in each connected dominating set. If exactly one vertex in the second to last row is dominating, then the vertex in the same column (and the last row) must be dominating and the other vertex in the last row can either be dominating or not. If the two vertices of the second to last row are dominating, then the vertices in the last row are dominated and therefore they can either be dominating or not. This yields the equation for the p_n.

It remains to prove the equations for the polynomials p'_n and p''_n. The idea is the same as before and Figure 6.5 illustrates the derivation of the two equations. The red vertices are the dominating vertices in the graph. $\qquad \square$
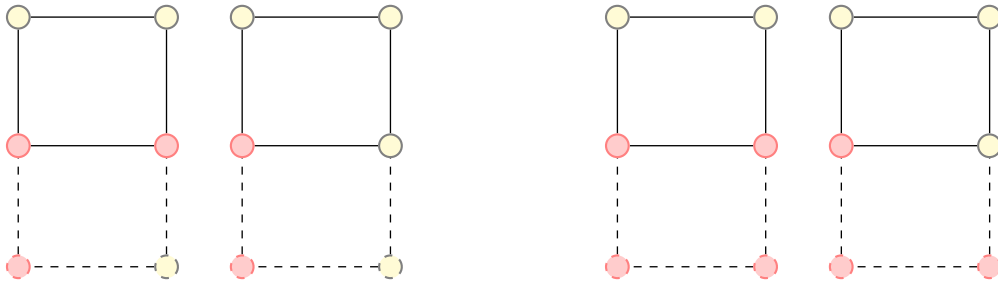
Fig. 6.5: Two possible situations for adding a row to p'_{n-1} (left) and to p''_{n-1} (right).

For the lexicographic product of two paths a nice theorem can be proved.

Theorem 6.55. *Let P_n and P_m be two paths, with $m, n \geq 3$. Then*

$$D_c(P_n \cdot P_m, x) = (1+x)^{2m}((1+x)^m - 1)^{n-2}.$$

Proof. In each row, beginning from the second up to the second to last row, at least one vertex has to be dominating to obtain a connected subset. These sets are also dominating sets in the whole graph and therefore the theorem follows. $\qquad \square$

6.4 Connected Domination Reliability Polynomial

Wireless sensor networks (WSNs) consist of small nodes with sensing, computation and wireless communications capabilities. They are widely used in many applications, including traffic control, geo-fencing of gas or oil pipelines, air pollution monitoring, and machine health monitoring (see [He12]). Suppose now that the nodes are subject to random failure, e.g. technical fault or energy issue, and the links are perfectly reliable. One may ask: What is the probability that in such a probabilistic WSN a set of connected operating nodes exist, such that every failed node is monitored? Such a set of nodes build a connected dominating set in the corresponding network.

Definition 6.56. *Let $G = (V, E)$ be a connected graph and the vertices of the graph are subject to random and independent failure with probability $q = 1 - p$. Then the connected domination reliability polynomial $\mathrm{DRel}_c(G, p)$ is defined as follows*

$$\mathrm{DRel}_c(G, p) = \sum_{k=1}^{n} d_k^c(G) p^k (1 - p)^{n-k}.$$

This reliability polynomial can be obtained from the connected domination polynomial.

Lemma 6.57. *Let $G = (V, E)$ be a graph and $\mathrm{D}_c(G, x)$ be the connected domination polynomial of G. Then the connected domination reliability polynomial can be calculated in the following way:*

$$\mathrm{DRel}_c(G, p) = (1 - p)^n \, \mathrm{D}_c(G, \frac{p}{1 - p}).$$

Proof. We simply use the definition of the connected domination reliability polynomial and perform some substitutions:

$$\begin{aligned}
\mathrm{DRel}_c(G, p) &= \sum_{k=1}^{n} d_k^c(G) p^k (1 - p)^{n-k} \\
&= (1 - p)^n \sum_{k=1}^{n} d_k^c(G) \left(\frac{p}{1 - p} \right)^k \\
&= (1 - p)^n \, \mathrm{D}_c(G, \frac{p}{1 - p}).
\end{aligned}$$

$\square$

Corollary 6.58. *Let $G = (V, E)$ be a graph and $\mathrm{DRel}_c(G, p)$ be the connected domination reliability polynomial. Then*

$$\mathrm{D}_c(G, x) = (1 + x)^n \, \mathrm{DRel}_c(G, \frac{x}{1 + x}).$$

The connected domination reliability polynomial yields a lower bound for the residual network reliability $R_1(G, p)$ (Definition 2.50) of the graph G (see Figure 6.6).

Lemma 6.59. *Let $G = (V, E)$ be a connected graph, $\mathrm{DRel}_c(G, p)$ be the connected domination reliability polynomial and $R_1(G, p)$ be the residual network reliability of G. Then*

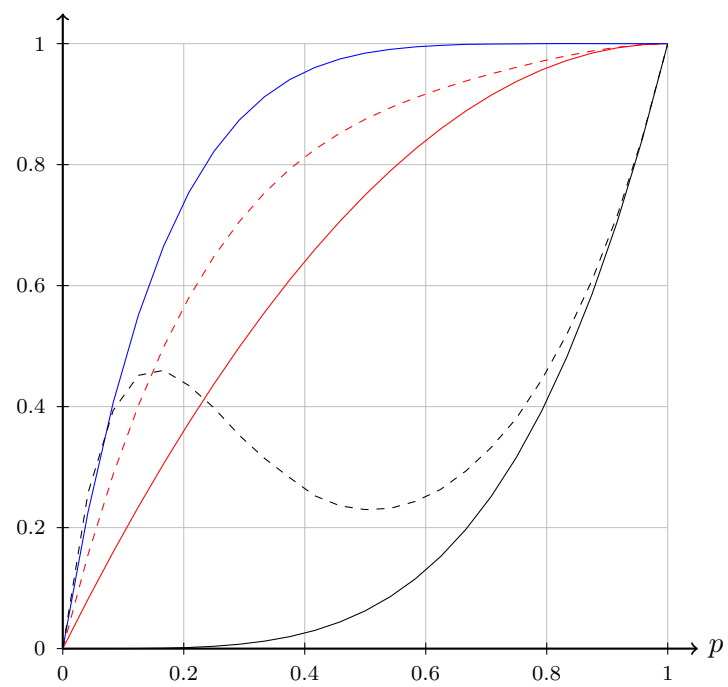$$\mathrm{DRel}_c(G, p) \leq R_1(G, p), \text{ for } 0 \leq p \leq 1.$$

Fig. 6.6: Reliability functions of the diamond graph (red), a random tree with 8 vertices and 4 leaves (black), the complete graph K_6 (blue), and the corresponding residual network reliability (dashed).

7 The Bipartition Polynomial

The domination polynomial exclusively counts dominating vertex subsets. A promising way to refine this polynomial is to encode the size of the neighborhood of a (not necessarily dominating) vertex subset W, e.g. $x^{|W|}y^{|N(W)|}$. Nevertheless, we call the vertices in such a vertex subset W dominating vertices. Let now ∂W be the set of all edges of G with exactly one of their end vertices in W. As a further generalization of the domination polynomial, we count the edges that are necessary in order to cover a subset of $N_G(W)$ which provides the following definition.

Definition 7.1. *The* bipartition polynomial *is defined in the following way:*

$$\mathrm{B}(G; x, y, z) = \sum_{W \subseteq V} x^{|W|} \sum_{F \subseteq \partial W} y^{|N_{G\langle F\rangle}(W)|} z^{|F|}.$$

It follows directly from the definition that the bipartition polynomial is multiplicative with respect to components of a graph. Let $G = (V, E)$ be a graph with two components G_1 and G_2, then

$$\mathrm{B}(G; x, y, z) = \mathrm{B}(G_1; x, y, z)\,\mathrm{B}(G_2; x, y, z).$$

The bipartition polynomial has some nice representations (see [Dod+15]). One of these representations is a multiplicative representation, which is useful for many proofs.

Theorem 7.2. *[Dod+15] The bipartition polynomial has the following multiplicative representation:*

$$B(G; x, y, z) = \sum_{W \subseteq V} x^{|W|} \prod_{v \in N_G(W)} \left(y\left((1+z)^{|N_G(v) \cap W|} - 1\right) + 1 \right). \tag{7.1}$$

Recurrence equations with respect to vertex or edge operations exist for many graph polynomials. In contrast, for the bipartition polynomial no such nice results can be proved.

Theorem 7.3. *Let $G = (V, E)$ be an arbitrary graph and v be a vertex of this graph. Then no linear recurrence equation with the operations $G - v$, G/v, $G - N[v]$, $G/N[v]$ and $G \odot v$ exists. More precisely, no rational functions $a, b, c, d, e \in \mathbb{R}(x, y, z)$ exists that satisfy*

$$\mathrm{B}(G; x, y, z) = a\,\mathrm{B}(G - v; x, y, z) + b\,\mathrm{B}(G/v; x, y, z) + c\,\mathrm{B}(G - N[v]; x, y, z)$$
$$+ d\,\mathrm{B}(G/N[v]; x, y, z) + e\,\mathrm{B}(G \odot v; x, y, z).$$

Proof. The proof uses the idea of the proof of Theorem 3.8. □

In spite of this result, it is sometimes possible to prove a recurrence equation if the graph has some special properties. The following theorem gives a result of this kind for the bipartition polynomial of a graph having a vertex of degree one that is adjacent to a degree-two vertex.

Theorem 7.4. *Let $G = (V, E)$ be a graph, $u, w \in V$, $e = \{u, w\} \in E$, $deg_G(u) = 1$ and $v \notin V$. Then the bipartition polynomial satisfies:*

$$\mathrm{B}(G + v + \{u, v\}; x, y, z) = (1 + x)\,\mathrm{B}(G; x, y, z) + xyz(2 + z)\,\mathrm{B}(G - u; x, y, z)$$
$$- xyz^2(1 - y)\,\mathrm{B}(G\dagger e; x, y, z).$$

Proof. The bipartition polynomial can be split into three parts with respect to the vertex u. The first part $\mathrm{B}_1^u(G; x, y, z)$ is the domination polynomial under the assumption that the vertex u is dominating. The second part $\mathrm{B}_2^u(G; x, y, z)$ is the polynomial under the assumption that u is non-dominating, but u is dominated in G. The last case is that u is not a dominating vertex and is non-dominated in G. Observe that the sum of these three polynomials is the bipartition polynomial $\mathrm{B}(G; x, y, z)$ of the graph G. We obtain:

$$\mathrm{B}_1^u(G; x, y, z) = \sum_{\substack{W \subseteq V \\ u \in W}} x^{|W|} \sum_{F \subseteq \partial W} y^{|N_{(V,F)}(W)|} z^{|F|}, \tag{7.2}$$

$$\mathrm{B}_2^u(G; x, y, z) = \sum_{\substack{W \subseteq V \\ u \notin W}} x^{|W|} \sum_{\substack{F \subseteq \partial W \\ u \in N_{(V,F)}(W)}} y^{|N_{(V,F)}(W)|} z^{|F|}, \tag{7.3}$$

$$\mathrm{B}_3^u(G; x, y, z) = \sum_{\substack{W \subseteq V \\ u \notin W}} x^{|W|} \sum_{\substack{F \subseteq \partial W \\ u \notin N_{(V,F)}(W)}} y^{|N_{(V,F)}(W)|} z^{|F|}. \tag{7.4}$$

For the sake of convenience we write $\mathrm{B}_i^u(G)$ instead of $\mathrm{B}_i^u(G; x, y, z)$, for $i \in \{1, 2, 3\}$. Considering the neighbor $w \in V$ of u, we can simplify the three polynomials $\mathrm{B}_1^u(G)$, $\mathrm{B}_2^u(G)$ and $\mathrm{B}_3^u(G)$. If u is a dominating vertex, we have three possible cases for the vertex w. If w is a dominating vertex, the edge $\{u, w\}$ has no influence on the polynomial. If w is dominated in $G - u$, we can either add the edge $\{u, w\}$ or leave it out. This leads to $(1 + z)\,\mathrm{B}_2^w(G - u)$. If w is non-dominated, then we can either include the edge in the selected edge subset F, so that w will be dominated by u, or we leave the edge out. This leads to $(1 + yz)\,\mathrm{B}_3^w(G - u)$. With these considerations we obtain:

$$\mathrm{B}_1^u(G) = \sum_{\substack{W \subseteq V \\ u \in W}} x^{|W|} \sum_{F \subseteq \partial W} y^{|N_{(V,F)}(W)|} z^{|F|}$$
$$= x\Big(\mathrm{B}_1^w(G - u) + (1 + z)\mathrm{B}_2^w(G - u) + (1 + yz)\mathrm{B}_3^w(G - u)\Big). \tag{7.5}$$

In the second case, u is dominated by w in G. So the vertex w must be dominating and the edge $\{u, w\}$ must be counted. Due to the fact that $deg(u) = 1$, we can remove the vertex u and the edge $\{u, w\}$ in the two sums.

$$\mathrm{B}_2^u(G) = \sum_{\substack{W \subseteq V \\ u \notin W}} x^{|W|} \sum_{\substack{F \subseteq \partial W \\ u \in N_{(V,F)}(W)}} y^{|N_{(V,F)}(W)|} z^{|F|}$$
$$= \sum_{W \subseteq V \setminus \{u\}} x^{|W|} \sum_{\substack{F \subseteq \partial W \\ u \in N_{(V,F)}(W)}} y^{|N_{(V,F)}(W)|} z^{|F|}$$
$$= yz \sum_{\substack{W \subseteq V \setminus \{u\} \\ w \in W}} x^{|W|} \sum_{F \subseteq \partial W \setminus \{w, u\}} y^{|N_{(V,F)}(W)|} z^{|F|}$$
$$= yz\mathrm{B}_1^w(G - u). \tag{7.6}$$

In the third case, u is not a dominating vertex and it is non-dominated. This means that we can remove the vertex u and the edge $\{u, w\}$ from the two sums and calculate the polynomial in $G - u$.

$$
\begin{aligned}
\mathrm{B}_3^u(G) &= \sum_{\substack{W \subseteq V \\ u \notin W}} x^{|W|} \sum_{\substack{F \subseteq \partial W \\ u \notin N_{(V,F)}(W)}} y^{|N_{(V,F)}(W)|} z^{|F|} \\
&= \sum_{W \subseteq V \backslash \{u\}} x^{|W|} \sum_{F \subseteq \partial W \backslash \{w,u\}} y^{|N_{(V,F)}(W)|} z^{|F|} \\
&= \mathrm{B}(G - u; x, y, z).
\end{aligned}
\tag{7.7}
$$

If we now add the vertex v and the edge $\{u, v\}$ to the graph G, we can write the bipartition polynomial as follows:

$$
\begin{aligned}
\mathrm{B}(G + v + \{u, v\}; x, y, z) = {}&(1 + x)\,\mathrm{B}(G; x, y, z) \\
&+ z\left(y\mathrm{B}_1^u(G) + x\mathrm{B}_2^u(G) + xy\mathrm{B}_3^u(G)\right).
\end{aligned}
\tag{7.8}
$$

If the edge $\{u, v\}$ is not used (or more precisely, it is not counted in the second sum of the bipartition polynomial), then the vertex v can be dominating or not. This gives us the first part of the sum. If the edge is used (either u or v is dominating), the vertex u can be in three states. If u is a dominating vertex, then the new vertex v will be dominated by u. So we obtain $yz\mathrm{B}_1^u(G)$. If u in G is dominated, then v must be a dominating vertex. This leads to $xz\mathrm{B}_2^u(G)$. If u is non-dominating and is non-dominated, then v must be a dominating vertex. This leads to the last part $xyz\mathrm{B}_3^u(G)$.

The substitution of (7.5), (7.6) and (7.7) in Equation (7.8) yields:

$$
\begin{aligned}
\mathrm{B}(G + v + \{u, v\}; x, y, z) = {}&(1 + x)\,\mathrm{B}(G; x, y, z) \\
&+ z\Big[yx\Big(\mathrm{B}_1^w(G - u) + (1 + z)\mathrm{B}_2^w(G - u) \\
&\quad + (1 + yz)\mathrm{B}_3^w(G - u)\Big) \\
&\quad + xyz\mathrm{B}_1^w(G - u) + xy\,\mathrm{B}(G - u; x, y, z)\Big] \\
= {}&(1 + x)\,\mathrm{B}(G; x, y, z) \\
&+ z\Big[yx\Big(\mathrm{B}(G - u; x, y, z) + z\mathrm{B}_2^w(G - u) + yz\mathrm{B}_3^w(G - u)\Big) \\
&\quad + xyz\mathrm{B}_1^w(G - u) + xy\,\mathrm{B}(G - u; x, y, z)\Big] \\
= {}&(1 + x)\,\mathrm{B}(G; x, y, z) \\
&+ z\Big[2xy\,\mathrm{B}(G - u; x, y, z) \\
&\quad + xyz\left(\mathrm{B}_1^w(G - u) + \mathrm{B}_2^w(G - u) + y\mathrm{B}_3^w(G - u)\right)\Big] \\
= {}&(1 + x)\,\mathrm{B}(G; x, y, z) \\
&+ z\Big[(2xy + xyz)\,\mathrm{B}(G - u; x, y, z) - xyz(1 - y)\mathrm{B}_3^w(G - u)\Big] \\
= {}&(1 + x)\,\mathrm{B}(G; x, y, z) \\
&+ z\Big[(2xy + xyz)\,\mathrm{B}(G - u; x, y, z) \\
&\quad - xyz(1 - y)\,\mathrm{B}(G - u - w; x, y, z)\Big]
\end{aligned}
$$

$$= (1 + x)\, \mathrm{B}(G; x, y, z)$$
$$+ xyz(2 + z)\, \mathrm{B}(G - u; x, y, z) - xyz^2(1 - y)\, \mathrm{B}(G{\dagger}e; x, y, z).$$

$\square$

Corollary 7.5. *From Theorem 7.4 we directly obtain a recurrence equation for the path P_n $(n \geq 2)$:*

$$\mathrm{B}(P_{n+1}; x, y, z) = (1 + x)\, \mathrm{B}(P_n; x, y, z) + xyz(2 + z)\, \mathrm{B}(P_{n-1}; x, y, z)$$
$$- xyz^2(1 - y)\, \mathrm{B}(P_{n-2}; x, y, z).$$

The initial conditions are

$$\mathrm{B}(P_0; x, y, z) = 1,$$
$$\mathrm{B}(P_1; x, y, z) = 1 + x,$$
$$\mathrm{B}(P_2; x, y, z) = (1 + x)^2 + 2xyz.$$

Let $B(G; x, y, z)$ be the bipartition polynomial of G. Then we define

$$b_{ik}(G) = [x^i y^k] B(G; x, y, z)$$

and

$$b_{ikl}(G) = [x^i y^k z^l] B(G; x, y, z).$$

Theorem 7.6. *Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be graphs, $n = |V(G_1)|$ and $m = |V(G_2)|$. Then*

$$\mathrm{B}(G_1 * G_2; x, y, z) = \sum_{k=0}^{n} \sum_{l=0}^{n-k} \sum_{i=0}^{m} \sum_{j=0}^{m-i} x^{k+i} y^{l+j} b_{kl}(G_1) b_{ij}(G_2)(1 + z)^{kj+il}$$
$$(y((1 + z)^k - 1) + 1)^{m-i-j} (y((1 + z)^i - 1) + 1)^{n-k-l}.$$

Proof. Suppose that $W_i \subseteq V(G_i)$, $Y_i \subseteq V(G_i)$ and $Z_i = V(G_i) \backslash W_i \backslash Y_i$ with $W_i \cap Y_i = \emptyset$, for $i \in \{1, 2\}$. Furthermore, let $|W_1| = k$, $|Y_1| = l$, $|W_2| = i$ and $|Y_2| = j$. Suppose that the vertices in W_i are the dominating vertices in G_i and the vertices in Y_i are the vertices dominated by W_i. The polynomial $b_{kl}(G_1) b_{ij}(G_2)$ counts the possibilities to choose such pairs of vertex subsets and the edges between them. In $G_1 * G_2$ we have to count the possibilities to choose edges between W_1 and Y_2 and W_2 and Y_1. Therefore, we obtain the term $(1 + z)^{kj+il}$. Additionally, the vertices in Z_2 can be dominated by vertices in W_1 and the vertices in Z_1 by vertices in W_2, respectively. If we choose at least one edge between a vertex of Z_2 and an arbitrary vertex of W_1, the vertices in Z_2 are dominated. This is counted by $(y((1 + z)^k - 1) + 1)^{m-i-j}$ and yields the theorem. $\square$

Theorem 7.7. *Let $G = (V(G), E(G))$ be a simple graph and $n = |V(G)|$. Then*

$$\mathrm{B}(G * K_1; x, y, z) = x(1 + yz)^n\, \mathrm{B}\left(G; \frac{x}{1 + yz}, \frac{y(1 + z)}{1 + yz}, z\right)$$
$$+ y\, \mathrm{B}(G; x(1 + z), y, z) - (y - 1)\, \mathrm{B}(G; x, y, z).$$

Proof. Let $K_1 = (\{v\}, \emptyset)$ be a graph with one vertex. If the vertex v is a dominating vertex, then v dominates all vertices in $G * K_1$ that are not already dominated. Additionally, we can choose edges between v and vertices in G which are already dominated from other vertices in G. This yields the first term of the sum:

$$x \sum_{W \subseteq V(G)} x^{|W|} \sum_{F \subseteq \partial W} y^{|N_{(V,F)}(W)|} (1+yz)^{n-|W|-|N_{(V,F)}(W)|} (1+z)^{|N_{(V,F)}(W)|} z^{|F|}$$

$$= x(1+yz)^n \sum_{W \subseteq V(G)} \left(\frac{x}{1+yz} \right)^{|W|} \sum_{F \subseteq \partial W} \left(\frac{y(1+z)}{1+yz} \right)^{|N_{(V,F)}(W)|} z^{|F|}.$$

If the vertex v is dominated, then we can choose an edge from every dominating vertex in G to the vertex v, but at least one edge must be chosen. This leads to the second term in the sum:

$$y \sum_{W \subseteq V(G)} x^{|W|} \left((1+z)^{|W|} - 1 \right) \sum_{F \subseteq \partial W} y^{|N_{(V,F)}(W)|} z^{|F|}.$$

If v is neither dominating nor dominated, then no edge between v and G occurs and we must simply add $\mathrm{B}(G; x, y, z)$, which yields the theorem. $\qquad\square$

Corollary 7.8. *Let $F_n = (V, E)$ be a fan with n vertices. Then*

$$\begin{aligned}\mathrm{B}(F_n; x, y, z) = {} & x(1+yz)^n \, \mathrm{B}(P_{n-1}; \frac{x}{1+yz}, \frac{y(1+z)}{1+yz}, z) \\ & + y \, \mathrm{B}(P_{n-1}; x(1+z), y, z) - (y-1) \, \mathrm{B}(P_{n-1}; x, y, z).\end{aligned}$$

Corollary 7.9. *Let $W_n = (V, E)$ be a wheel with n vertices. Then*

$$\begin{aligned}\mathrm{B}(W_n; x, y, z) = {} & x(1+yz)^n \, \mathrm{B}(C_{n-1}; \frac{x}{1+yz}, \frac{y(1+z)}{1+yz}, z) \\ & + y \, \mathrm{B}(C_{n-1}; x(1+z), y, z) - (y-1) \, \mathrm{B}(C_{n-1}; x, y, z).\end{aligned}$$

7.1 Encoded Graph Invariants

From the bipartition polynomial a lot of graph invariants can be obtained. Such numeric invariants are the order of the graph G

$$|V(G)| = \deg(B(G; x, 1, 1))$$

and its size

$$|E(G)| = \frac{1}{2}[xyz]B(G, x, y, z).$$

The maximum size of an edge cut of G is given by

$$c_{\max}(G) = \deg(B(G, 1, 1, z))$$

which provides the number of maximum edge cuts

$$\left[z^{c_{\max}(G)} \right] B(G, 1, 1, z).$$

Theorem 7.10. *Let $G = (V, E)$ be a simple graph with the bipartition polynomial $\mathrm{B}(G; x, y, z)$. Then*

$$\delta(G) = \min\{i | [xz^i]\, \mathrm{B}(G; x, 1, z - 1) > 0\},$$
$$\Delta(G) = \max\{i | [xz^i]\, \mathrm{B}(G; x, 1, z - 1) > 0\}, \text{ and}$$
$$\sharp \deg_i(G) = [xz^i]\, \mathrm{B}(G; x, 1, z - 1).$$

Proof. Substitute y with 1 and z with $z - 1$ in Equation (7.1). This yields

$$B(G; x, 1, z - 1) = \sum_{W \subseteq V} x^{|W|} \prod_{v \in N_G(W)} z^{|N_G(v) \cap W|}.$$

Hence, the coefficient of xz^i in $B(G; x, 1, z - 1)$ yields the number of vertices with exactly i neighbors and the theorem follows. $\qquad\square$

Corollary 7.11. *The number of isolated vertices $\mathrm{iso}(G)$ of the graph $G = (V, E)$ can be determined by*

$$\mathrm{iso}(G) = [x]\, \mathrm{B}(G; x, 1, -1).$$

Moreover, the last results show that the *degree generating function* of a graph $G = (V, E)$ is

$$\sum_{v \in V} t^{\deg v} = [x^1]\, B(G; x, 1, t - 1).$$

Theorem 7.12. *Let $G = (V, E)$ be a graph with m edges. Then*

$$s_k = \delta_k [x^k z^{m - \binom{k}{2} - \binom{n-k}{2}}] z^m\, \mathrm{B}(G; x, 1, z - 1)$$

yields the number of splittings of the graph in two cliques with k and $n - k$ vertices, with

$$\delta_k = \begin{cases} \frac{1}{2}, & \text{if } k = \frac{n}{2} \\ 1, & \text{otherwise.} \end{cases}$$

Proof. The coefficient of x^k in the polynomial $z^m\, \mathrm{B}(G; x, 1, \frac{1}{z} - 1)$ yields the number of edges in the graph which are not between the k dominating vertices and the corresponding dominated vertices. If the two sets, the dominating vertices W and the other vertices in $V - W$, are cliques, then there must be $m - \binom{k}{2} - \binom{n-k}{2}$ edges between the two sets. If $k = n/2$, then we count every set of two cliques twice. $\qquad\square$

Corollary 7.13. *Let c_l be the number of l-cliques in the graph $G = (V, E)$. Then*

$$c_{n-1} = [xz^{\binom{n-1}{2}}] z^m\, \mathrm{B}(G; x, 1, \frac{1}{z} - 1).$$

The bipartition polynomial encodes a variety of graph polynomials (see [Dod+15] and Figure 9.1 on page 129). The next theorem shows the connection to the domination polynomial.

Theorem 7.14. *Let $G = (V, E)$ be a graph, $D(G, x)$ its domination polynomial and $\mathrm{B}(G; x, y, z)$ its bipartition polynomial. Then*

$$D(G, x) = [y^n]\, \mathrm{B}(G; xy, 1 - y, -1).$$

Proof. Observe that we have $N(v) \cap W \neq \emptyset$, for all $v \in N(W)$. Now we substitute x with xy, y with $1 - y$ and z with -1 in Equation (7.1). This yields

$$B(G; xy, 1 - y, -1) = \sum_{W \subseteq V} (xy)^{|W|} \prod_{v \in N(W)} \left[(1 - y) \left[0^{|N(v) \cap W|} - 1 \right] + 1 \right]$$

$$= \sum_{W \subseteq V} (xy)^{|W|} \prod_{v \in N(W)} y$$

$$= \sum_{W \subseteq V} x^{|W|} y^{|W| + |N(W)|}.$$

All vertex subsets W with $|W| + |N(W)| = n$ are dominating sets and the theorem follows. $\square$

Definition 7.15. *Let $G = (V, E)$ be a graph. Then the* matching polynomial *of G is the ordinary generating function*

$$\mu(G, x) = \sum_i m_i(G) x^i.$$

The coefficient m_k of the polynomial is the number of matchings of size k.

Remark 7.16. *Sometimes the matching polynomial is defined as*

$$\mu(G, x) = \sum_i (-1)^i m_i(G) x^{n-2i}.$$

Theorem 7.17. *Let $G = (V, E)$ be a graph and $\mu(G, x)$ be the matching polynomial of G. Then*

$$\mu(G, x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{x}{2} \right)^k \sum_{i=1}^{k} (-1)^{k-i} \binom{n-k-i}{k-i} b_{ikk}(G).$$

Proof. Let G be a graph of order n and let k be a given positive integer. We use the abbreviations $b_i = b_{ikk}(G)$ and $p = n - k$. Let $D_i(G)$ be the set of all bipartite subgraphs $H = (X \cup Y, F)$ of G with $|X| = i$, $|Y| = |F| = k$, such that all vertices of Y have degree 1 in H. Hence the cardinality of $D_i(G)$ is b_i. Observe that we consider $X \cup Y$ as an *ordered* bipartition, which implies that bipartite subgraphs which are identical except that the sets X and Y are exchanged are counted twice in b_i. Let $C_i(G)$ be the subset of $D_i(G)$ consisting of those (ordered) bipartite subgraphs of G that do not have any isolated vertices in X and define $c_i = |C_i(G)|$. As the end vertices of any edge in a matching can be arbitrarily assigned to X or Y, we have $2^k m_k = c_k$. Each bipartite subgraph in $D_i(G)$ that contains exactly $i - j$ isolated vertices in X consists of a subgraph H from $C_j(G)$ and a selection of $i - j$ vertices out of the $n - k - j$ vertices that do not belong to H. Consequently, we obtain

$$b_i = \sum_{j=1}^{i} \binom{n-k-j}{i-j} c_j$$

$$= \sum_{j=1}^{i} \binom{p-j}{i-j} c_j. \qquad (7.9)$$

The theorem states that

$$m_k = \frac{1}{2^k} \sum_{i=1}^{k} (-1)^{k-i} \binom{n-k-i}{k-i} b_{ikk}(G)$$

and hence

$$c_k = \sum_{i=1}^{k} (-1)^{k-i} \binom{p-i}{k-i} b_i.$$

Replacing k by j and substituting c_j in (7.9) yields

$$b_i = \sum_{j=1}^{i} \binom{p-j}{i-j} \sum_{l=1}^{j} (-1)^{j-l} \binom{p-l}{j-l} b_l$$

$$= \sum_l b_l \sum_j \binom{p-l}{j-l} \binom{p-j}{i-j} (-1)^{j-l}.$$

Thus it remains to prove that

$$\sum_j \binom{p-l}{j-l} \binom{p-j}{i-j} (-1)^{j-l} = \delta_{il}.$$

Rearranging the binomial coefficients yields

$$\sum_j \binom{p-l}{j-l} \binom{p-j}{i-j} (-1)^{j-l} = \binom{p-l}{p-i} \sum_j \binom{i-l}{i-j} (-1)^{j-l}.$$

If $i = l$, then the last sum has only one non-vanishing term, which is 1. Otherwise, if $i \neq l$, then the binomial coefficient or the sum vanishes, which completes the proof. $\square$

The following theorems show connections of the bipartition polynomial to other graph polynomials. These results can be found in [Dod+15].

Theorem 7.18. *[Dod+15] Let $\mathrm{B}(G; x, y, z)$ be the bipartition polynomial and $\mathrm{C}(G; z)$ be the cut generating function of the graph G. Then*

$$\mathrm{C}(G; z) = \frac{1}{2} \mathrm{B}(G; 1, 1, z-1).$$

Peter Tittmann [DT15] defined the *extended cut polynomial*

$$J(G; x, y) = \sum_{W \subseteq V} x^{|W|} y^{|E(W)|+|\partial W|}.$$

This polynomial can also be obtained from the bipartition polynomial in r-regular graphs.

Theorem 7.19. *Let $G = (V, E)$ be a r-regular graph and $J(G; x, y)$ its extended cut polynomial. Then*

$$J(G; x, y) = B(G; x\sqrt{y^r}, 1, \sqrt{y} - 1).$$

Proof. If G is a r-regular graph, then $r|W| = 2|E(W)| + |\partial W|$ and therefore $|E(W)| = 1/2(r|W| - |\partial W|)$. Using Equation (7.1) we can verify that

$$B(G; xt^r, 1, z-1) = \sum_{W \subseteq V} x^{|W|} t^{r|W|} z^{|\partial W|}.$$

Now replacing t with $\sqrt{y^r}$ and z with $\frac{y}{\sqrt{y}}$ yields:

$$\sum_{W \subseteq V} x^{|W|} \sqrt{y^{r|W|}} \left(\frac{y}{\sqrt{y}} \right)^{|\partial W|} = \sum_{W \subseteq V} x^{|W|} y^{1/2(r|W|-|\partial W|)} y^{|\partial W|}$$
$$= \sum_{W \subseteq V} x^{|W|} y^{|E(W)|} y^{|\partial W|}$$

which proves the theorem. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let $G = (V, E)$ be a graph, then the *Ising polynomial* $Z(G; x, y)$ is defined as follows

$$Z(G; x, y) = x^n y^m \sum_{W \subseteq V} x^{-|W|} y^{-|\partial W|}.$$

Theorem 7.20. *[Dod+15] The Ising polynomial of a graph $G = (V, E)$ with n vertices and m edges is given by*

$$Z(G; x, y) = x^n y^m B\left(G; \frac{1}{x}, 1, \frac{1}{y} - 1 \right).$$

Theorem 7.21. *[Dod+15] Let $G = (V, E)$ be a simple r-regular graph. Then the independence polynomial of G is given by*

$$I(G, t) = \lim_{x \to 0} B\left(G; tx^r, 1, \frac{1}{x} - 1 \right).$$

The great variety of graph invariants encoded in the bipartition polynomial leads to the question how well this polynomial distinguishes non-isomorphic graphs. Figure 7.1 shows the smallest pair of non-isomorphic graphs with the same bipartition polynomial. These graphs were presented in [Mar14] as an example for non-isomorphic graphs with the same Potts model partition function. We could show by exhaustive computer search that all non-isomorphic trees with up to 15 vertices and all non-isomorphic graphs with up to 9 vertices can be distinguished by their bipartition polynomial.
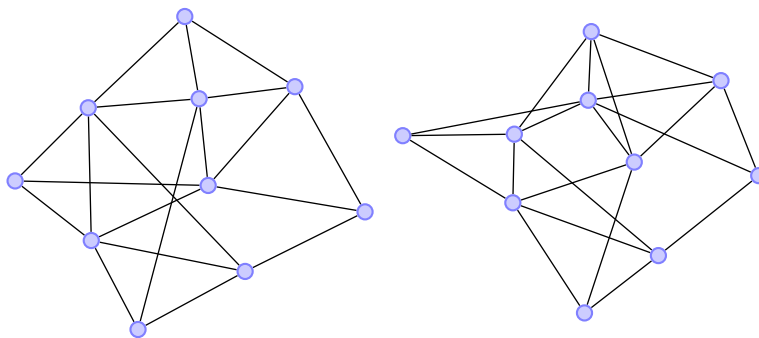
Fig. 7.1: The smallest pair of non-isomorphic graphs with the same bipartition polynomial.

7.2 Special Graph Classes

In this section we investigate the bipartition polynomial of some special graph classes. It is obvious that the bipartition polynomial of the edgeless graph E_n is simply $(1 + x)^n$. But other graph classes are more interesting. In this section we show some results for complete and nearly complete graphs, and substitution graphs.

7.2.1 Complete and Nearly Complete Graphs

Theorem 7.22. *Let K_n be a complete graph with n vertices. Then*

$$\mathrm{B}(K_n; x, y, z) = \sum_{k=0}^{n} \binom{n}{k} x^k (y(1+z)^k - y + 1)^{n-k}.$$

Proof. Any vertex subset W of cardinality k in K_n has an open neighborhood of size $n - k$ and each vertex v of this neighborhood has exactly k edges that link v with a vertex in W. We obtain

$$B(K_n; x, y, z) = \sum_{k=0}^{n} \binom{n}{k} x^k \sum_{j=0}^{n-k} \binom{n-k}{j} y^j ((1+z)^k - 1)^j.$$

Here $(1+z)^k - 1$ is the ordinary generating function for the choice of a nonempty subset of the set of k edges that connect v with a vertex in W. Now the theorem follows by simplification of the inner sum using the binomial theorem. $\qquad\square$

With Theorem 7.7 we can calculate the bipartition polynomial for nearly complete graphs. Let $G = (V, E)$ be a nearly complete graph and let $C \subset V$ be a maximum clique in G. In the first step we calculate the bipartition polynomial of $G - C$ and then add successively the vertices of the clique C using Theorem 7.7.

In the following theorems we will give some explicit equations for some special nearly complete graphs.

Theorem 7.23. *Let n be an even number greater than or equal to two and let $G = (V, E)$ be a graph obtained from the complete graph K_n by removing a perfect matching. Then*

$$\mathrm{B}(G; x, y, z) = \sum_{k=0}^{\frac{n}{2}} \binom{\frac{n}{2}}{k} x^{2k} \sum_{i=0}^{\frac{n}{2}-k} \binom{\frac{n}{2}-k}{i} 2^i x^i \left(y \left(1+z\right)^{2k+i-1} - y + 1 \right)^i$$

$$\left(y \left(1+z\right)^{2k+i} - y + 1 \right)^{n-2k-2i}$$

Proof. If we remove a perfect matching in the complete graph, then there are $n/2$ pairs of non-adjacent vertices. In the theorem, we count the number of such pairs where the two vertices are dominated by k and by i the number of pairs where exactly one vertex is dominating. In the second case we can change the roles of the two vertices of the pair and get 2^i possible sets. Let now $W \subseteq V$ be a vertex subset of the graph, $v \in W$, $u \notin W$ and $|W| = 2k + i$. Then u has $2k + i - 1$ neighbors in W. All other vertices, which are not in W, have $2k + i$ neighbors in W. Together with the ideas of the proof of the Theorem 7.22 the theorem follows. $\qquad\square$

It is also possible to calculate the bipartition polynomial of an (n, k)-star.

Theorem 7.24. *Let $G = (V, E)$ be an (n, k)-star. Then*

$$\mathrm{B}(S_{n,k}; x, y, z) = \sum_{i=0}^{k} \binom{k}{i} x^i \sum_{j=0}^{n-k} \binom{n-k}{j} x^j$$

$$(y(1+z)^i - y + 1)^{n-k-j} (y(1+z)^{i+j} - y + 1)^{k-i}.$$

Proof. Let V_2 be the vertices of the k-clique and $V_1 = V - V_2$. First of all, we choose i vertices from V_2 and j vertices from V_1. Now we sum over all possibilities to choose vertices in V_2 which are non-dominating. All of these vertices can be dominated by a vertex in V_1. Now we choose l vertices from V_1 which are non-dominating. These vertices can be dominated by the dominating vertices in V_1 and V_2. With these considerations we obtain

$$\mathrm{B}(S_{n,k}; x, y, z) = \sum_{i=0}^{k} \binom{k}{i} x^i \sum_{j=0}^{n-k} \binom{n-k}{j} x^j$$
$$\sum_{b=0}^{n-k-j} \binom{n-k-j}{b} y^b ((1+z)^i - 1)^b \sum_{l=0}^{k-i} \binom{k-i}{l} y^l ((1+z)^{i+j} - 1)^l.$$

Now the theorem follows by simplification of the inner two sums using the binomial theorem. $\qquad\square$

Remark 7.25. *The approach presented in Theorem 7.24 can be generalized in order to find the bipartition polynomial of a split graph.*

7.2.2 Substitution Graphs

Let $H = (V(H), E(H))$ be a simple graph and let $G = (V(G), E(G))$ be a graph with a distinguished vertex u. The graph $H^{G,u}$ is obtained from H by gluing a copy of G at the vertex u on each vertex v of H. For the following theorem and its proof, we use again the polynomials B_i^u, $i \in \{1, 2, 3\}$, introduced in the proof of Theorem 7.4 (see Equations (7.2), (7.3), and (7.4)).

Theorem 7.26. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be graphs and u be a vertex of G.*

$$\mathrm{B}(H^{G,u}; x, y, z) = (\mathrm{B}(G - u; x, y, z) + \mathrm{B}_2^u(G))^{|V(H)|}$$
$$\mathrm{B}\left(H, \frac{\mathrm{B}_1^u(G)}{\mathrm{B}(G - u) + \mathrm{B}_2^u(G)}, \frac{y\,\mathrm{B}(G - u) + \mathrm{B}_2^u(G)}{\mathrm{B}(G - u) + \mathrm{B}_2^u(G)}, z\right).$$

Proof. First we consider a term $\alpha_{ijkl} t^i x^j y^k z^l$ of the expanded form of the following polynomial:

$$f(H) = \sum_{W \subseteq V(H)} x^{|W|} \sum_{F \subseteq \partial W} y^{|N_{(V,F)}(W)|} t^{|V| - |N_{(V,F)}[W]|} z^{|F|}.$$

The coefficient α_{ijkl} counts set quadruples (T, W, S, F) with $|T| = i$, $|W| = j$, $|S| = k$, $|F| = l$, such that T, W, S are disjoint subsets of $V(H)$ with $T \cup S \cup W = V(H)$ and $F \subseteq E(H)$. The set W is the set of dominating vertices and S comprises all dominated vertices that are non-dominating. All edges of F link a vertex of W with a vertex of S. If a vertex u is in W, it is also a dominating vertex in the attached copy of the graph G. So we can substitute x in $f(H)$ by $\mathrm{B}_1^u(G)$. If a vertex u is in S, then it can either be dominated or not in the corresponding copy of G. The term $\mathrm{B}_2^u(G) + y\,\mathrm{B}_3^u(G)$ yields exactly the polynomial for this situation and we substitute y in $f(H)$ with it. The rest of the vertices, counted by t, must have the same state in the attached copy of G. So we substitute t with $\mathrm{B}_3^u(G)$. Observe that $\mathrm{B}_3^u(G) = \mathrm{B}(G - u; x, y, z)$. This argumentation leads to the following formula, which yields the theorem.

$$B(H^{G,u}; x, y, z) = \sum_{W \subseteq V(H)} B_1^u(G)^{|W|} \sum_{F \subseteq \partial_H W} (B_2^u(G) + y B_3^u(G))^{|N_{(V,F)}(W)|}$$

$$B_3^u(G)^{|V(H)| - |N_{(V,F)}[W]|} z^{|F|}$$

$$= (B_2^u(G) + B_3^u(G))^{|V(H)|} \sum_{W \subseteq V(H)} \left(\frac{B_1^u(G)}{B_3^u(G) + B_2^u(G)} \right)^{|W|}$$

$$\sum_{F \subseteq \partial_H W} \left(\frac{B_2^u(G) + y B_3^u(G)}{B_3^u(G) + B_2^u(G)} \right)^{|N_{(V,F)}(W)|} z^{|F|}.$$

$\square$

Let S_n be a star with n vertices and let $v \in V$ be the center vertex of S_n. Then we can split the bipartition polynomial in three parts with respect to the vertex v:

$$S_n^1 = B_1^v(S_n) = x(1 + x + yz)^{n-1},$$
$$S_n^2 = B_2^v(S_n) = y \left((1 + x(1 + z))^{n-1} - (1 + x)^{n-1} \right),$$
$$S_n^3 = B_3^v(S_n) = (1 + x)^{n-1}.$$

Moreover, let $S_n^{2,3} = S_n^2 + S_n^3$.

A *hedgehog* is a graph H_n with $2n$ vertices such that n vertices induce a clique in H_n and each vertex of the clique is adjacent to exactly one vertex outside of the clique. Consequently, the edge set of H_n can be partitioned into the edge set of a n-clique and the edge set of a perfect matching of size n. A *generalized hedgehog* $H(G)$ is a graph, obtained from a graph G by attaching a pending edge to each vertex of G. A *star-hedgehog* $S_k(G)$ is a graph obtained from a graph G by attaching k pendent edges to each vertex of G (see Figure 7.2).
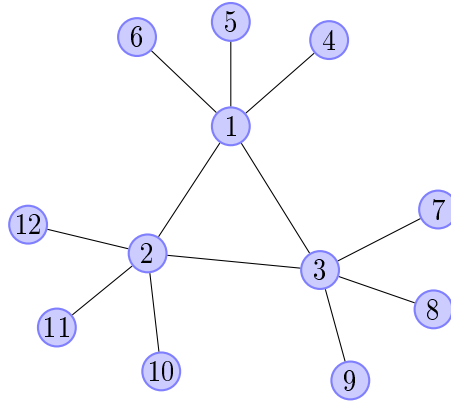

Fig. 7.2: Star-hedgehog with K_3 as center

Corollary 7.27. *Let $G = (V, E)$ be a graph with n vertices and $k \geq 1$. Then*

$$B(S_k(G); x, y, z) = \left(y(xz + x + 1)^k - (y-1)(x+1)^k \right)^{|V(G)|}$$

$$B\left(G; \frac{S_{k+1}^1}{S_{k+1}^{2,3}}, \frac{y S_{k+1}^3 + S_{k+1}^2}{S_{k+1}^{2,3}}, z \right).$$

7.3 Counting Bipartite Subgraphs

First, we introduce the graph polynomial $\tilde{B}(G, z)$ which counts the number of bipartite subgraphs with respect to the number of the edges of these subgraphs. Let $G = (V, E)$ be a simple graph, then

$$\tilde{B}(G, z) = \sum_{\substack{F \subseteq E \\ (V,F) \text{ is bipartite}}} z^{|F|}.$$

The bipartition polynomial can also be represented as a sum over the bipartite subgraphs of a graph G.

Theorem 7.28. *[Dod+15] The bipartition polynomial satisfies*

$$B(G; x, y, z) = \sum_{\substack{F \subseteq E \\ (V,F) \text{ is bipartite}}} z^{|F|} (1 + x)^{\text{iso}(V,F)} \prod_{(V_1 \cup V_2, A) \in \text{Comp}(V,F)} (x^{|V_1|} y^{|V_2|} + x^{|V_2|} y^{|V_1|}),$$

where V_1 and V_2 are the bipartition sets of a covered component of (V, F) with the edge set A.

Thus the question arises: Is it possible to obtain the $\tilde{B}$-polynomial from the bipartition polynomial? This question is still open, even for bipartite graphs.

It follows directly from the definition that the $\tilde{B}$-polynomial is multiplicative in the components of a graph. Another interesting property of the $\tilde{B}$-polynomial is given by the next lemma.

Lemma 7.29. *Let $G = (V, E)$ be a bipartite graph. Then*

$$\sum_{F \subseteq E} (-1)^{|F|} \tilde{B}(G - F, z) = z^{|E|}.$$

Proof. Using the definition of the polynomial yields the lemma:

$$\sum_{F \subseteq E} (-1)^{|F|} \tilde{B}(G - F, z) = \sum_{F \subseteq E} (-1)^{|F|} \sum_{H \subseteq E - F} z^{|H|}$$

$$= \sum_{H \subseteq E} z^{|H|} \sum_{F \subseteq E - H} (-1)^{|F|}$$

$$= z^{|E|}.$$

$\square$

We can use the last lemma to prove the next statement.

Theorem 7.30. *Let $G = (V, E)$ be a connected graph. Then*

$$\sum_{F \subseteq E} (-1)^{|F|} \tilde{B}(G - F, z) = \begin{cases} z^{|E|}, & \text{if } G \text{ is bipartite} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. It remains to prove that the sum vanishes for non-bipartite graphs.

$$\sum_{F \subseteq E} (-1)^{|F|} \tilde{B}(G - F, z) = \sum_{F \subseteq E} (-1)^{|F|} \sum_{\substack{H \subseteq E - F \\ (V,H) \text{ bipartite}}} z^{|H|}$$

$$= \sum_{\substack{H \subseteq E \\ (V,H) \text{ bipartite}}} z^{|H|} \sum_{F \subseteq E - H} (-1)^{|F|}.$$

The second sum is only not equal to zero if H equals E, but this is not possible because of the fact that the graph G is non-bipartite. Therefore, the theorem follows. $\qquad \square$

Connection to the Edge-Cover Polynomial

In bipartite graphs the edge-cover polynomial (see Definition 2.45) can be calculated from the $\tilde{B}$-polynomials of the induced subgraphs.

Theorem 7.31. *Let $G = (V, E)$ be a bipartite graph and $E(G, z)$ be the edge-cover polynomial of G. Then*

$$E(G, z) = (-1)^{|V|} \left(\sum_{W \subseteq V} (-1)^{|W|} \tilde{B}(G[W], z) - 1 \right).$$

Proof. Again, we simply use the definition of the $\tilde{B}$-polynomial and obtain

$$\begin{aligned}
\sum_{W \subseteq V} (-1)^{|W|} \tilde{B}(G[W], z) &= \sum_{W \subseteq V} (-1)^{|W|} \sum_{F \subseteq G[W]} z^{|F|} \\
&= \sum_{F \subseteq E} z^{|F|} (-1)^{|\bigcup F|} \sum_{W \subseteq V \setminus \bigcup F} (-1)^{|W|} + 1 \\
&= (-1)^{|V|} \sum_{\substack{F \subseteq E \\ F \text{ is edge-cover}}} z^{|F|} + 1.
\end{aligned}$$

The second sum in the second to last line equals zero if at least one vertex is not covered by the chosen edges and therefore only edge-covers of the graph will be counted. $\qquad \square$

Theorem 7.32. *Let $G = (V, E)$ be a graph. Then*

$$\sum_{F \subseteq E} (-1)^{|F|} E(G\langle F \rangle, z) = (-z)^{|E|}.$$

Proof. Using the definition of the edge-cover polynomial yields:

$$\begin{aligned}
\sum_{F \subseteq E} (-1)^{|F|} E(G\langle F \rangle, z) &= \sum_{F \subseteq E} (-1)^{|F|} \sum_{\substack{H \subseteq F \\ H \text{ is edge-cover in } G\langle F \rangle}} z^{|H|} \\
&= \sum_{\substack{H \subseteq E \\ H \text{ is edge-cover}}} z^{|H|} \sum_{F \supseteq H} (-1)^{|F|} \\
&= \sum_{\substack{H \subseteq E \\ H \text{ is edge-cover}}} z^{|H|} (-1)^{|H|} \sum_{F \subseteq E \setminus H} (-1)^{|F|} \\
&= (-z)^{|E|}.
\end{aligned}$$

$\qquad \square$

Theorem 7.33. *Let $G = (V, E)$ be a graph. Then*

$$\sum_{F \subseteq E} (-1)^{|F|} E(G - F, z) = z^{|E|}.$$

Proof. Using the definition of the edge-cover polynomial yields:

$$\sum_{F \subseteq E} (-1)^{|F|} E(G - F, z) = \sum_{F \subseteq E} (-1)^{|F|} \sum_{\substack{H \subseteq E \setminus F \\ \text{H is edge-cover}}} z^{|H|}$$

$$= \sum_{\substack{H \subseteq E \\ \text{H is edge-cover}}} z^{|H|} \sum_{F \subseteq E \setminus H} (-1)^{|F|}$$

$$= z^{|E|}.$$

$\square$

Theorem 7.34. *Let* $G = (V, E)$ *be a graph. Then*

$$\sum_{W \subseteq V} (-1)^{|W|} E(G[W], z) = \sum_{F \subseteq E} (-1)^{|\bigcup F|} z^{|F|}.$$

Proof. Again using the definition of the polynomial yields:

$$\sum_{W \subseteq V} (-1)^{|W|} E(G[W], z) = \sum_{W \subseteq V} (-1)^{|W|} \sum_{\substack{F \subseteq E(G[W]) \\ \text{F is edge-cover in } G[W]}} z^{|F|}$$

$$= \sum_{F \subseteq E} z^{|F|} \sum_{\substack{W \subseteq V \\ \text{F is edge-cover in } G[W]}} (-1)^{|W|}$$

$$= \sum_{F \subseteq E} (-1)^{|\bigcup F|} z^{|F|}.$$

Every edge subset of the graph has exactly one corresponding vertex subset in which this edge subset is vertex-cover, namely the set which consists of the end vertices of the edges. $\square$

Definition 7.35. *Let* $G = (V, E)$ *be a graph with random failing edges. The edges are assumed to fail independently with identical probability* $q = 1 - p$. *Then the probability that the graph has no isolated vertex is denoted by* $\mathrm{P}_{\mathrm{iso}}(G, p)$.

Lemma 7.36. *Let* $G = (V, E)$ *be a graph with random failing edges. Then*

$$\mathrm{P}_{\mathrm{iso}}(G, p) = (1 - p)^{|E|} E(G, p/(1 - p)).$$

Proof. A spanning subgraph $G\langle F \rangle$, with $F \subseteq E$, has no isolated vertex if and only if F is an edge-cover in G. Therefore,

$$(1 - p)^{|E|} E(G, p/(1 - p)) = \sum_{\substack{F \subseteq E \\ \text{F is edge-cover}}} p^{|F|} (1 - p)^{|E| - |F|}$$

yields the probability that the graph has no isolated vertex. $\square$

Theorem 7.37. *[AO13] Let $G = (V, E)$ be a graph with m edges and no isolated vertex. Furthermore, let $e_i(G)$ be the number of edge-covers with i edges in G and*

$$\tilde{e}_i(G) = \binom{m}{i} - \sum_{v \in V} \binom{m - d(v)}{i}.$$

Then

$$e_i(G) \geq \tilde{e}_i(G), \quad \forall i \in \{1, \ldots, m - 2\delta + 1\}$$
$$e_i(G) = \tilde{e}_i(G), \quad \forall i \in \{m - 2\delta + 2, m\}.$$

We can use this theorem to obtain a lower bound for $\mathrm{P}_{\mathrm{iso}}(G, p)$.

Corollary 7.38. *Let $G = (V, E)$ be a graph with m edges and no isolated vertex. Then for all $p \in [0, 1]$*

$$\mathrm{P}_{\mathrm{iso}}(G, p) \geq \sum_{i=1}^{m} [\tilde{e}_i(G) \geq 0] \tilde{e}_i(G) p^i (1 - p)^{m-i}.$$

8 Three Possible Generalizations of the Domination Polynomial

The generalization of counting polynomials is a promising way for a better understanding of these polynomials and it shows interesting connections to other polynomials. In this chapter three generalizations will be presented. The aim of these definitions is to show some possible directions for further research.

8.1 The General Domination Polynomial

Definition 8.1. *Let $G = (V, E)$ be a graph. Then the* general domination polynomial *is defined as*

$$\mathrm{E}(G; x, y, z, w) = \sum_{W \subseteq V} x^{|W|} y^{|N(W)|} z^{\mathrm{iso}(G[W])} w^{c(G[W])}.$$

The general domination polynomial has some connections to other graph polynomials.

Theorem 8.2. *Let $G = (V, E)$ be a graph, $\mathrm{Q}(G; v, x)$ be the subgraph component polynomial, $\mathrm{Y}(G; x, y, z)$ be the trivariate domination polynomial and $\mathrm{D}_c(G, x)$ be the connected domination polynomial of G. Then*

$$\mathrm{Q}(G; v, x) = \mathrm{E}(G; v, 1, x, x),$$
$$\mathrm{Y}(G; x, y, z) = \mathrm{E}(G; x, y, z, 1),$$
$$\mathrm{D}_c(G, x) = [y^n z]\, \mathrm{E}(G; xy, y, z, z).$$

Proof. Using the definition of the general domination polynomial yields

$$\mathrm{E}(G; v, 1, x, x) = \sum_{W \subseteq V} v^{|W|} x^{\mathrm{iso}(G[W])} x^{c(G[W])}$$
$$= \sum_{W \subseteq V} v^{|W|} x^{k(G[W])} = \mathrm{Q}(G; v, x)$$
$$\mathrm{E}(G; x, y, z, 1) = \sum_{W \subseteq V} x^{|W|} y^{|N(W)|} z^{\mathrm{iso}(G[W])}$$
$$= \mathrm{Y}(G; x, y, z).$$

The coefficient of $x^i y^j z^k$ of the polynomial

$$\mathrm{E}(G; xy, y, z, z) = \sum_{W \subseteq V} x^{|W|} y^{|N[W]|} z^{k(G[W])}$$

counts the number of sets W of size i that dominate $j - i$ other vertices and the subgraphs $G[W]$ having k components. The coefficient of $y^n z$ yields the generating function of the connected dominating sets of the graph. $\qquad\square$

Corollary 8.3. *Let $G = (V, E)$ be a forest with n vertices. Then*

$$\chi(G, x) = [v^n] \operatorname{E}(G; v(x-1), 1, x/(x-1), x/(x-1)).$$

Proof. This corollary follows immediately from the connection of the general domination polynomial and the subgraph component polynomial. Trinks [Tri12b] showed that the subgraph component polynomial and the subgraph counting polynomial are equivalent if the graph is a forest. This yields the connection to the chromatic polynomial. $\qquad\square$

Theorem 8.4. *Let $G = (V, E)$ be a graph which vertices fail independently of each other with the probability $q = 1 - p$. Furthermore, let $R_1(G, p)$ and $R_k(G, p)$ be the residual network reliability and the k-residual network reliability, respectively. Then*

$$R_1(G, p) = (1-p)^n [z] \operatorname{E}(G; p/(1-p), 1, z, z) \text{ and}$$

$$R_k(G, p) = (1-p)^n \sum_{i=k}^{n} [x^i z] \operatorname{E}(G; xp/(1-p), 1, z, z).$$

Proof. Substitution of the variables in the definition of the general domination polynomial yields

$$(1-p)^n \operatorname{E}(G; p/(1-p), 1, z, z) = (1-p)^n \sum_{W \subseteq V} (p/(1-p))^{|W|} z^{\operatorname{iso}(G[W]) + c(G[W])}$$

$$= \sum_{W \subseteq V} p^{|W|} (1-p)^{n-|W|} z^{k(G[W])}.$$

The coefficient of z^1 yields the desired polynomial. The proof of the second equation is analogous. $\qquad\square$

As a result of the connection of $\operatorname{E}(G; x, y, z, w)$ to the trivariate domination polynomial all results from Section 5.5.1 are valid for the general domination polynomial. The already known results can be found in the following list:

- The order of the graph: $|V(G)| = \deg_v \operatorname{E}(G; v, 1, x, x)$ [TAM11].

- The size of the graph: $|E(G)| = [v^2 x] \operatorname{E}(G; v, 1, x, x)$ [TAM11].

- The number of the components: $k(G) = \deg_x(v^{|V(G)|} \operatorname{E}(G; v, 1, x, x))$ [TAM11].

- The size of the components (see Theorem 5.41).

- The independence number: $\alpha(G) = \deg_x \operatorname{E}(G; v, 1, x, x)$ [TAM11].

- The number of isolated vertices (see Theorem 5.41).

- The degree generating function (see Theorem 5.41).

- The number of induced C_4 and P_4 if the graph is a k-regular bipartite graph [LH13].

For some special graph classes, it is possible to prove explicit equations with respect to the number of vertices of the graph. Every vertex subset of size k in the edgeless graph E_n yields the term $x^k z^k$ and therefore

$$\operatorname{E}(E_n; x, y, z, w) = (1 + xz)^n.$$

In the complete graph every vertex subset of size greater or equal two has exactly one covered component and the remaining vertices are in the neighborhood of this vertex subset. If the vertex subset is of size one, then it is an isolated vertex and again all other $n-1$ vertices are in the neighborhood. This yields

$$E(K_n; x, y, z, w) = y^n w((1 + x/y)^n - nx/y - 1) + nxy^{n-1}z + 1.$$

Let now $G = (V_1 \cup V_2, E)$ be a complete bipartite graph, $|V_1| = m$ and $|V_2| = n$. If at least one vertex of V_2 and no vertex of V_1 is chosen, then all vertices of V_1 are dominated. This yields the term $y^m((1 + xz)^n - 1)$. The same argumentation holds if at least one vertex of V_1 and no vertex of V_2 is chosen. If from both of the two sets at least one vertex is chosen, then this vertex subset is connected and all vertices that are not in this subset are dominated. Hence, we obtain

$$E(K_{m,n}; x, y, z, w) = y^n \Big((1 + xz)^m - 1) + wy^m((1 + x/y)^m - 1)((1 + x/y)^n - 1) \Big)$$
$$+ y^m((1 + xz)^n - 1) + 1.$$

Distinguishing Non-Isomorphic Graphs

The two non-isomorphic graphs in Figure (5.6) have the same general domination polynomial and moreover, they are the smallest pair of non-isomorphic graphs with the same general domination polynomial. The pair is also an example for two non-isomorphic graphs with different Tutte polynomials but equal subgraph component polynomials. Liao and Hou [LH13] showed that the subgraph component polynomial cannot distinguish between some non-isomorphic graphs that can be distinguished by the characteristic polynomial, the matching polynomial and the Tutte polynomial (see Figure 8.1). Nevertheless, these special graphs G_1 and G_2 are distinguishable by the general domination polynomial.
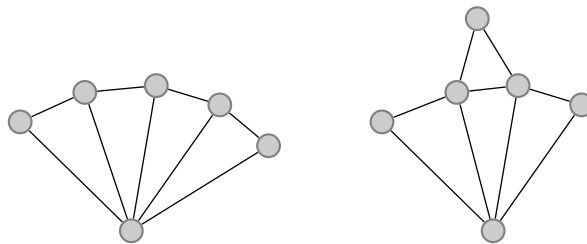

Fig. 8.1: The graphs F_6 and F_5^-.

Let F_n be the fan graph and F_n^+ arises from F_n by adding a new vertex v and edges between v and a degree-two vertex and its adjacent degree-three vertex of the fan (Figure 8.2).
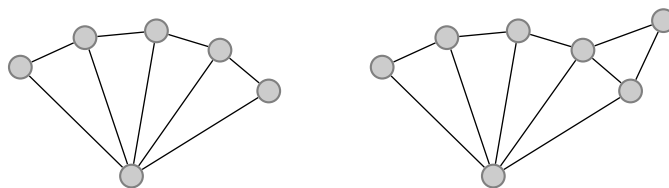

Fig. 8.2: The graphs F_6 and F_6^+.

Theorem 8.5. *[LH13] Let* $\mathrm{T}(G, x)$ *be the Tutte polynomial of the graph* G. *Then for* $n \geq 5$

$$\mathrm{T}(F_n, x) = \mathrm{T}(F_{n-1}^+, x) \text{ and}$$
$$\mathrm{Q}(F_n; v, x) \neq \mathrm{Q}(F_{n-1}^+; v, x).$$

The last theorem together with Theorem 8.2 yields (for $n \geq 5$)

$$\mathrm{E}(F_n; x, y, z, w) \neq \mathrm{E}(F_{n-1}^+; x, y, z, w).$$

8.2 The General Bipartition Polynomial and the Most General Domination Polynomial

In this section we introduce two possible generalizations, one of the general domination polynomial and one of the bipartition polynomial. The first generalization yields a connection between the bipartition polynomial and the trivariate domination polynomial. We call it the general bipartition polynomial $\bar{E}$.

Definition 8.6. *Let* $G = (V, E)$ *be a graph. Then the* general bipartition polynomial *is defined by*

$$\bar{E}(G; w, x, y, z) = \sum_{W \subseteq V} x^{|W|} w^{\mathrm{iso}(G[W])} \sum_{F \subseteq \delta W} y^{|N_{(V,F)}(W)|} z^{|F|}.$$

The connection of the general bipartition polynomial to the bipartition polynomial and the trivariate domination polynomial follows immediately from the definition:

$$\mathrm{B}(G; x, y, z) = \bar{E}(G; 1, x, y, z),$$
$$\mathrm{Y}(G; x, y, z) = \bar{E}(G; z, x, 1 - y, -1).$$

The most general domination polynomial generalizes all domination-related graph polynomials investigated in this thesis.

Definition 8.7. *Let* $G = (V, E)$ *be a graph. Then the* most general domination polynomial *is defined by*

$$\tilde{E}(G; v, w, x, y, z) = \sum_{W \subseteq V} x^{|W|} v^{c(G[W])} w^{\mathrm{iso}(G[W])} \sum_{F \subseteq \delta W} y^{|N_{(V,F)}(W)|} z^{|F|}.$$

One can easily see the connection to the general bipartition polynomial and the general domination polynomial. One possible direction of further research is to investigate the connections of the general bipartition polynomial and the most general domination polynomial to other known graph polynomials. It also remains to prove recurrence equations of these polynomials.

9 Conclusions and Open Problems

In this thesis several theorems for different graph polynomials are proved. The main results deal with the domination-related polynomials, especially proofs of recurrence equations, different representations and the calculation in special graph classes. We also defined some new counting polynomials, like the connected domination polynomial. Furthermore, one main result of the thesis is the definition of the trivariate domination polynomial and the proofs of some properties of this polynomial. Moreover, several results about the calculation of the graph polynomials in product graphs are proved. We also showed the connection of the studied counting polynomials to the corresponding reliability polynomials.

The two generalizations expatiated in this thesis (the bipartition polynomial and the trivariate domination polynomial) have many nice properties and connections to other graph polynomials. Figure 9.1 shows the connection between different graph polynomials. This figure is an extension of the "graph of graph polynomials" presented by M. Trinks [Tri12a]. In his thesis one can find more information and properties of the polynomials in the figure. The arrows in this figure illustrate the connection between the graph polynomials. In the case of a dashed arrow, the connection exists only for some special graph classes.

Open Problems

In this thesis we proved some results for the different polynomials, but there are a lot of open questions. In the following a selection of these problems is given.

For the independent domination polynomial of the tensor product of two paths only the combinations $P_n \times P_2$ and $P_n \times P_3$ are known (see Section 4.3.2). Is there a way to generalize these results to $P_n \times P_m$, $m \geq 4$, or to prove the given results in an easier way? Concerning the Cartesian product the question arises, whether is it possible to find a faster way to calculate the (independent/total/connected) domination polynomial of the product $G \square H$?

In fact, little is known about the domination related graph polynomials of strong products. Can the known results for this product be extended to $G \boxtimes P_n$ or $G \boxtimes C_n$?

The domination polynomial can be calculated with the essential sets of a graph (i-essential sets for the independent domination polynomial and t-essential sets for the total domination polynomial). This leads directly to the next question.

Problem 9.1. *How many essential, t-essential and i-essential sets does a given graph have? Is it possible to prove upper and lower bounds for these numbers?*

In Section 5.5.1, some connections from the trivariate domination polynomial to other graph polynomials and some encoded graph invariants are shown. But it seems that the trivariate domination polynomial has some more connections, especially to other neighborhood related polynomials.

Problem 9.2.

1. *Are there (more) connections between the trivariate domination polynomial and non-neighborhood related polynomials?*

2. Are there more graph invariants encoded in the trivariate domination polynomial?

In Section 5.5.4 some Y-unique graph classes are shown. This leads to the following problem.

Problem 9.3. *Are there more graph classes which can be distinguished by the trivariate domination polynomial?*

Some interesting open questions remain for future research with respect to the bipartition polynomial. One interesting question is, whether the bipartition polynomial has more connections to non-domination-related graph polynomials. Especially, is there any connection to coloring-related polynomials, like the chromatic polynomial or the Tutte polynomial?

The bipartition polynomial distinguishes non-isomorphic graphs quite well. However, it seems to be difficult to characterize non-isomorphic graphs that have the same bipartition polynomial.

Problem 9.4. *Which properties of two non-isomorphic graphs cannot be distinguished by the bipartition polynomial?*

It might be of interest for further research to study the domination problems in hypergraphs. We assume that the definition of the bipartition polynomial and the trivariate domination polynomial can be easily extended to hypergraphs. But especially for the independent or the total domination polynomial careful considerations will be necessary.
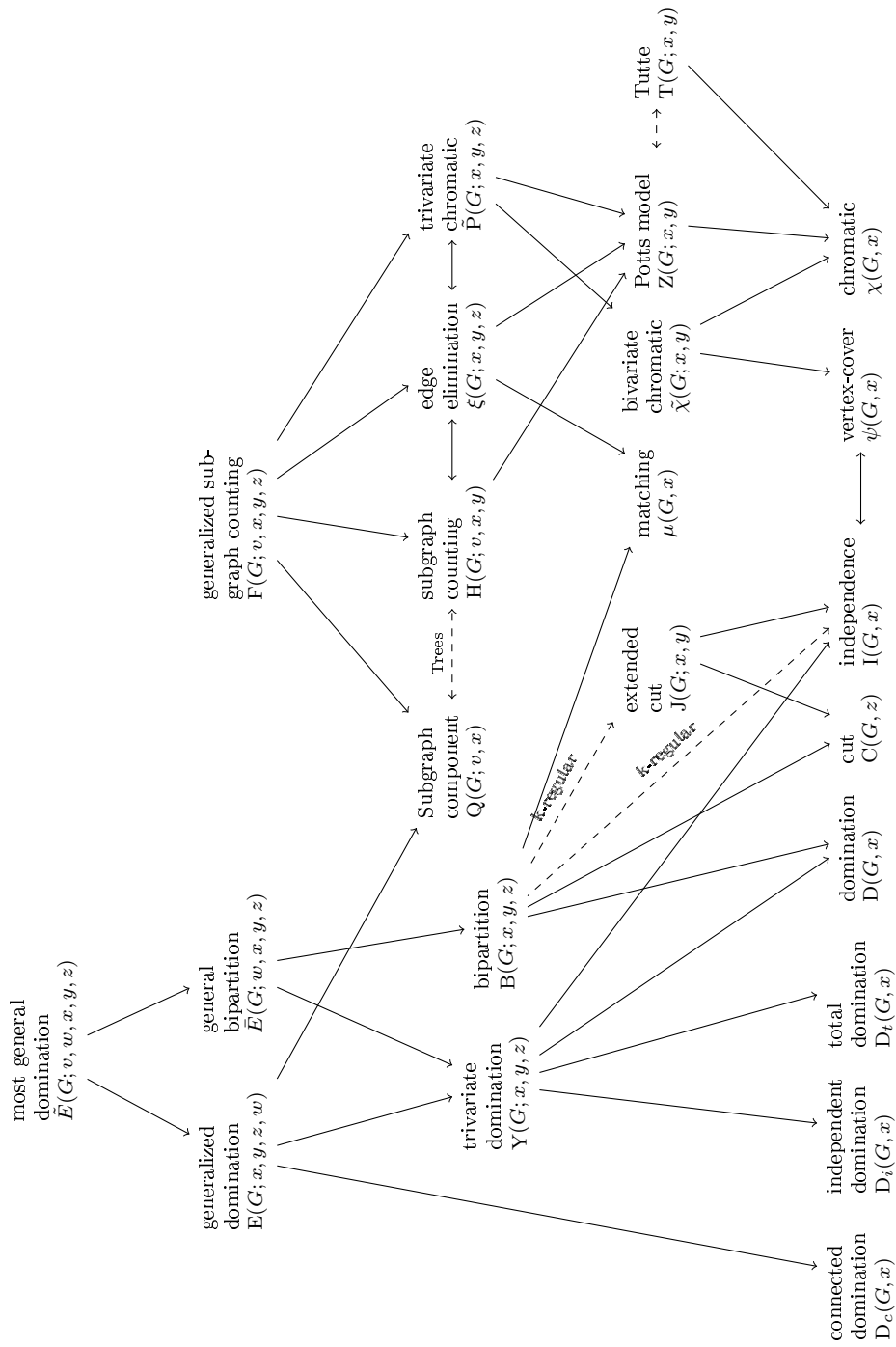
Fig. 9.1: Graph of graph polynomials.

Notation

Symbol	Description	Def.	Page
$N(v)$	Open neighborhood of the vertex v	2	17
$N[v]$	Closed neighborhood of the vertex v	2	17
$N(W)$	Open neighborhood of the vertex subset $W \subseteq V$	2	17
$N^t(W)$	Total open neighborhood of the vertex subset $W \subseteq V$	2	17
$N[W]$	Closed neighborhood of the vertex subset $W \subseteq V$	2	17
$PN(u, U)$	Set of the private neighbors of u with respect to the vertex subset U	2	17
$\deg_G(v)$	Degree of the vertex v	2	17
$\delta(G)$	Minimum degree of G	2	17
$\Delta(G)$	Maximum degree of G	2	17
$\sharp \deg_i(G)$	Number of vertices with degree i of G	2	17
$\bar{G}$	The complement of G	2	17
$L(G)$	The line graph of G	2	17
$G\langle F \rangle$	Spanning subgraph ($F \subseteq E$)	2.3	18
$G[U]$	Induced subgraph ($U \subseteq V$)	2.4	18
$E(U)$	Set of edges which are completely inside U ($U \subseteq V$)	2.4	18
$G[F]$	Edge-induced subgraph ($F \subseteq E$)	2.6	18
$k(G)$	Number of components of G	2.7	18
$c(G)$	Number of covered components of G	2.7	18
$\text{iso}(G)$	Number of isolated vertices of G	2.7	18
$\text{Comp}(G)$	Set of covered components of G	2.7	18
λ_G	The type of G	2.8	18
$\alpha(G)$	Independence number of G	2	18
$\omega(G)$	Clique number of G	2	18
(G_1, G_2, X)	Splitting of G	2.10	19
$G - v$	Deletion of the vertex v	2.1	19
G/v	Neighborhood completion of the vertex v and deletion of v	2.1	19
$G \odot v$	Removal of all edges between vertices of $N(v)$	2.1	19
$G \circledcirc v$	Graph $(G \odot v) - v$	2.1	19
$G \circ v$	Removal of v and adding loops to all neighbors of v	2.1	19
$G - X$	Removal of all vertices in X	2.1	19
$G\backslash_x X$	Fusion of the vertex subset X (the new vertex is labeled with x).	2.1	19
$G \rhd\!\!-\!\circ X$	Fusion of the vertex subset X and adding a new vertex which is adjacent to the fused one.	2.1	19
$G + \{v, \cdot\}$	Adding a new adjacent vertex to v	2.1	19
$G + \{X, \cdot\}_u$	Adding a new vertex u and edges joining all vertices of X with u.	2.1	19
$G - e$	Deletion of the edge e	2.1	19

Symbol	Description	Def.	Page
G/e	Contraction of the edge e	2.1	19
$G\dagger e$	Extraction of the edge $e = \{u,v\}$ $(G-u-v)$	2.1	19
K_n	Complete graph with n vertices	2.2	20
E_n	Edgeless graph with n vertices	2.2	20
K_n^k	Simple k-bounded complete graph.	2.14	20
$\Lambda(K_n^k)$	Type of the simple k-bounded complete graph K_n^k	2.14	20
$K_{m,n}$	Complete bipartite graph	2.2	20
$K_{m,n}^k$	k-bounded bipartite graph	2.15	20
$S_{n,k}$	(n,k)-star	2.19	21
$P_n^{(k)}$	Simple k-path with n vertices	2.24	22
$C_n^{(k)}$	k-cycle with n vertices	2.25	22
$G\square H$	The Cartesian product of G and H	2.27	23
$G \times H$	The tensor product of G and H	2.29	24
$G \cdot H$	The lexicographic product of G and H	2.31	25
$G \boxtimes H$	The strong product of G and H	2.33	25
$G \cup H$	The union of G and H	2.3	26
$G * H$	The join of G and H	2.35	26
F_n	Fan with n vertices	2.36	26
W_n	Wheel with n vertices	2.36	26
S_n	Star with n vertices	2.36	26
$G \circ H$	Corona graph of G and H	2.37	26
$\chi(G,x)$	Chromatic polynomial of G	2.40	27
$\mathrm{D}(G,x)$	Domination polynomial of G	2.41	27
$\Psi(G,x)$	Vertex-cover polynomial of G	2.42	27
$I(G,x)$	Independence polynomial of G	2.43	28
$E(G,z)$	Edge-cover polynomial of G	2.45	28
$\mathrm{Q}(G;v,x)$	Subgraph component polynomial of G	2.46	28
$\mathrm{R}(G;x,y)$	Rank polynomial of G	2.47	28
$\mathrm{T}(G;x,y)$	Tutte polynomial of G	2.48	28
$R_1(G,p)$	Residual network reliability of G	2.50	28
$R_k(G,p)$	k-residual network reliability of G	2.50	28
$\gamma(G)$	Domination number of G	3	31
$\mathrm{N}(G,x)$	The neighborhood polynomial	3.2	31
$\mathrm{DRel}(G,p)$	Domination reliability polynomial of G	3.3	41
$\mathrm{EDRel}(G,A,p)$	Edge failure domination reliability polynomial of G	3.32	42
$\gamma_i(G)$	Independent dominating number of G	4	43
$d_k^i(G)$	Number of independent dominating sets of size k	4.1	43
$\mathrm{D}_i(G,x)$	The independent domination polynomial of G	4.1	43
$\exp(G,r)$	R-expansion of G	4.6	44
$\mathrm{Ess}_i(G)$	Set of the i-essential sets of G	4.13	46
Cen_n	The centipede with $2n$ vertices.	4.4.1	61
$\mathrm{DRel}_i(G,p)$	Independent domination reliability polynomial of G	4.64	64
$\mathrm{d}_t(G)$	Number of the total dominating sets of G	5	67
$d_k^t(G)$	Number of total dominating sets of size k	5.2	67
$\mathrm{D}_t(G,x)$	The total domination polynomial of G	5.2	67
$\mathrm{Con}(G)$	Set of all vertex-induced conformal subgraphs	5	68

Symbol	Description	Def.	Page
$\mathrm{Ess}_t(G)$	Family of t-essential sets of G	5.10	70
$\mathrm{DRel}_t(G, p)$	The total domination reliability polynomial of G	5.32	77
$\mathrm{Y}(G; x, y, z)$	The trivariate domination polynomial of G	5.36	78
$\gamma_c(G)$	The connected domination number of G	6.2	91
$l(G)$	The maximum leaf number	6.3	91
$d_k^c(G)$	Number of connected dominating sets of size k	6.5	91
$\mathrm{D}_c(G, x)$	The connected domination polynomial of G	6.5	91
$\mathrm{DRel}_c(G, p)$	The connected domination reliability polynomial of G	6.56	104
$\mathrm{B}(G; x, y, z)$	The bipartition polynomial of G	7.1	107
$\mu(G, x)$	The matching polynomial of G	7.15	113
$J(G; x, y)$	The extended cut polynomial of G.	7.1	114
$Z(G; x, y)$	The Ising polynomial of G	7.1	115
$S_k(G)$	Star-hedgehog obtained from G	7.2.2	118
$\mathrm{P}_{\mathrm{iso}}(G, p)$	Probability that the probabilistic graph G has no isolated vertex	7.35	121
$\mathrm{E}(G; x, y, z, w)$	The general domination polynomial of G	8.1	123
$\bar{E}(G; w, x, y, z)$	The general bipartition polynomial of G	8.6	126
$\tilde{E}(G; v, w, x, y, z)$	The most general domination polynomial of G	8.7	126

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Weitere Personen waren an der Abfassung der vorliegenden Arbeit nicht beteiligt.

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19. Oktober 2015   Markus Dod, M.Sc.

Declaration

I hereby declare that I completed this work without any improper help from a third party and without using any aids other than those cited. All ideas derived directly or indirectly from other sources are identified as such.

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I did not seek the help of a professional doctorate-consultant. Only those persons identified as having done so received any financial payment from me for any work done for me. This thesis has not previously been published in the same or a similar form in Germany or abroad.

19th October 2015   Markus Dod, M.Sc.