Seminar 1 / Metric Spaces

1. Are the terms $d(x,y)$ metric functions?
   a) $d(x,y) = \sin^2(x-y); \qquad x,y \in \mathbb{R}^1$
   b) $d(x,y) = \sqrt{|x-y|}; \qquad x,y \in \mathbb{R}^1$
   c) $d(x,y) = |\arctan(x-y)|\,; \qquad x,y \in \mathbb{R}^1$
   d) $d(x,y) = |x_1 - y_1|\,; \qquad x,y \in \mathbb{R}^2,\ x = (x_1,x_2)^T,\ y = (y_1,y_2)^T$

2. Verify the inequality:

$$\frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}; \qquad \forall a,b \in \mathbb{R}^1$$

   Tip: Use the monotony of the function $f(x) = \dfrac{x}{1+x}$.

3. Prove by using the inequality of number 2 that the set of all real sequences with the following function $d(x,y)$ is a metric space:

$$d(x,y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|x_k - y_k|}{1+|x_k - y_k|}; \qquad x = \{x_k\}_{k=1}^{\infty}\,; \quad y = \{y_k\}_{k=1}^{\infty}$$

4. Verify that the following two axioms are equivalent to the axioms of the metric space:
   a) $d(x,y) = 0 \Leftrightarrow x = y$
   b) $d(x,y) \leq d(x,z) + d(y,z) \quad \forall x,y,z \in \mathbb{X}$

5. Prove that the metric function is continuous, i.e.
   $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ imply $\lim_{n\to\infty} d(x_n,y_n) = d(x,y)$.

6. Let M be the set of all $n-$digit binary words $x = x_1 x_2 .... x_n$.
   The HEMMING - distance d_H of such two binary words x,y is given by the number of digits which are different between x and y.
   Verify that
   a) $d_H(x,y) = \sum_{k=1}^{n} [(x_k + y_k) \bmod 2]$
   b) (M, d_H) is a metric space.

7. Let M be the set of all sequences of natural numbers. The distance between two different elements $x = \{x_k\}_{k=1}^{\infty}$ and $y = \{y_k\}_{k=1}^{\infty}$ is defined by $1/\lambda$ such that λ is the smallest natural number satisfying $x_\lambda \neq y_\lambda$. Further $d(x,x) = 0$.
   Verify that (M,d) is a metric space. (It is an example of BAIRE Space.)

8. Let $(\mathbb{X}_1, d_1)$ and $(\mathbb{X}_2, d_2)$ be metric spaces. For any $x,y \in \mathbb{X}_1 \times \mathbb{X}_2$,
   $x = (x_1,x_2)$, $y = (y_1,y_2)$ the metric d is given by $d(x,y) = \max(d_1(x_1,y_1), d_2(x_2,y_2))$.
   Verify that $(\mathbb{X}_1 \times \mathbb{X}_2, d)$ is a metric space.

Seminar 2 / Open and Closed Sets

1. Let $(\mathbb{X}, d)$ be a metric space. A and B are proper subsets of $\mathbb{X}$: $A \subset \mathbb{X}, B \subset \mathbb{X}$.
   Prove that $A \subset B$ implies $A^+ \subseteq B^+$ and $\overline{A} \subseteq \overline{B}$.

2. Let $(\mathbb{X}, d)$ be a metric space. A and B are proper subsets of $\mathbb{X}$: $A \subset \mathbb{X}, B \subset \mathbb{X}$.
   Verify that: $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ and $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
   But $\overline{A \cap B} = \overline{A} \cap \overline{B}$ is not valid. Give a counterexample!

3. Give an example for the following facts:
   a) The intersection of an infinite collection of open sets must not be open.
   b) The union of an infinite collection of closed sets must not be closed.

4. Let $(\mathbb{X}, d)$ be a metric space. A and B are proper subsets of $\mathbb{X}$: $A \subset \mathbb{X}, B \subset \mathbb{X}$. Further let A be an open set and B a closed set. Verify that $A \setminus B$ is open and $B \setminus A$ is closed.

5. Write down the (derivated) set A^+, the set of all interior points $\overset{o}{A}$ and the closure $\overline{A}$ of the following sets $A \subset \mathbb{R}$:
   $a) \quad A = \left\{ \frac{(-1)^n n^2}{1+n} | n \in N \right\} \qquad c) \quad A = \bigcup_{n=1}^{\infty} \left[n - \frac{1}{n}; n + \frac{1}{2n} \right]$
   $b) \quad A = \bigcap_{n=1}^{\infty} \left[-\frac{1}{n}; 1 + \frac{1}{n} \right] \qquad d) \quad A = \bigcup_{n=1}^{\infty} \left(\frac{2^n - 1}{2^n}; \frac{2^{n+1} - 1}{2^{n+1}} \right)$

6. Let E be the set $E = \{0; \frac{1}{n}; \frac{1}{n} + \frac{1}{m} \mid n, m \in N\} \subset \mathbb{R}$ What is the (derivated) set E^+?

7. Let $(\mathbb{X}, d)$ be a metric space. F_1 and F_2 are closed proper subsets of $\mathbb{X}$: $F_1 \subset \mathbb{X}$, $F_2 \subset \mathbb{X}$ such that $F_1 \cap F_2 = \emptyset$. Prove that there exist open sets G_1 and G_2 such that $F_1 \subset G_1$, $F_2 \subset G_2$ and $G_1 \cap G_2 = \emptyset$.

8. Let $(\mathbb{X}, d)$ be a metric space. A is a proper subset of $\mathbb{X}$: $A \subset \mathbb{X}$, $\overline{A}$ is the closure of A. Let x be an interior point of $\overline{A}$. Do this imply that x is an interior point of A too?

9. ** Look for an example of a metric space $\mathbb{X}$ with the property, that there are more sets than the space $\mathbb{X}$ and the empty set which are both open and closed.

Seminar 3 / Completeness of Metric Spaces

1. Verify that the following spaces are complete:

    1. $\mathbb{X} = m$: space of all real bounded sequences such that $d(x,y) = \sup_i |x_i - y_i|,\ \sup_i |x_i| < \infty,\ \sup_i |y_i| < \infty.$
    2. $\mathbb{X} = c$: space of all convergent real sequences such that $d(x,y) = \sup_i |x_i - y_i|$.
    3. $\mathbb{X} = c_0$: space of all real null sequences such that $d(x,y) = \sup_i |x_i - y_i|$.

2. Is the set of all natural numbers together with the following metric a complete metric space?

    1. $d_1(m,n) = \frac{|m-n|}{m \cdot n}$
    2. $d_2(m,n) = \begin{cases} 0 & \text{for } m = n \\ 1 + \frac{1}{m+n} & \text{for } m \neq n \end{cases}$

3. Consider $\mathbb{R}^n$ with the metric $d(x,y) = \max_i |x_i - y_i|$. Prove:

    1. $(\mathbb{R}^n, d)$ is a metric space.
    2. $(\mathbb{R}^n, d)$ is complete.

4. Consider a metric space $(\mathbb{X}, d)$ and a proper subset $M \subset \mathbb{X}$. Let d_0 be the restriction of the metric d onto M. Prove:

    1. (M, d_0) is a metric space.
    2. If (M, d_0) is complete, then M is closed in $\mathbb{X}$.
    3. If $(\mathbb{X}, d)$ is complete, then follows: (M, d_0) is complete $\Leftrightarrow$ M is closed.

5. Consider a metric space $(\mathbb{X}, d)$ and a compact proper subset $A \subset \mathbb{X}$. Let f be a contiuous mapping $f: A \to \mathbb{R}$. Prove:

    1. $f(A)$ is a compact set.
    2. The function f has an absolute maximum and an absolute minimum on A.

6. *Let $C^1[a,b]$ be the set of all continuously differentiable functions with respect to $[a,b]$ For any $x(t),\ y(t) \in C^1[a,b]$ we define

$$d(x,y) = \max_{a \leq t \leq b} |x(t) - y(t)| + \max_{a \leq t \leq b} |x'(t) - y'(t)|.$$

    1. Verify that $C^1[a,b]$ is a complete metric space.
    2. Consider $C^m[a,b]$, the set of all m times continuously differentiable functions with respect to $[a,b]$. How can we define an analogous metric there?

Seminar 4 /Fix Point Theorem

1. Verify that the function $f : [0, \frac{1}{2}] \rightarrow \mathbb{R}$ such that $f(x) = x^3$ is a contractive mapping in the metric space $X = \mathbb{R}$ with $d(x, y) = |x - y|$.

2. Verify that the function $f : [a, b] \rightarrow [c, d]$ such that $[c, d] \subseteq [a, b]$ and $|f'(x)| \leq \alpha < 1$ in $[a, b]$ is a contractive mapping in the metric space $X = \mathbb{R}$ with $d(x, y) = |x - y|$.

3. Look for the numbers $\lambda \in (0; 4]$ such that the mapping $f(x) = \lambda x(1 - x)$ with $0 \leq x \leq 1$ is a contractive mapping in the metric space $X = \mathbb{R}$ with $d(x, y) = |x - y|$.

4. Let $a_{ik} \in \mathbb{C},\ i, k = 1, 2, ..., n$ be the coefficients of the following system of linear equations

$$\begin{aligned} x_i - \sum_{k=1}^{n} a_{ik} x_k &= b_i \qquad i = 1, 2, ..., n \qquad \text{with} \\ \max_{1 \leq i \leq n} \sum_{k=1}^{n} |a_{ik}| &\leq q < 1. \end{aligned}$$

Show that this system of linear equations has a unique solution for every $b_1, ...b_n \in \mathbb{C}$.

Seminar 5 / Normed Spaces

1. Let $\mathbb{U}$ be a complete normed space and $\mathbb{S}$ be a proper subspace . Prove: The closure $\overline{\mathbb{S}}$ of $\mathbb{S}$ is a subspace of $\mathbb{U}$ too.

2. Let $(\mathbb{U}_1, \|.\|_1)$ and $(\mathbb{U}_2, \|.\|_2)$ be normed spaces over the field $\mathbb{K}$. Verify:
a) $\mathbb{U}_1 \times \mathbb{U}_2$ is a normed space with $\|(x_1, x_2)\| = \|x_1\|_1 + \|x_2\|_2$ for every $(x_1, x_2) \in \mathbb{U}_1 \times \mathbb{U}_2$.
b) If $\mathbb{U}_1$ and $\mathbb{U}_2$ are Banach spaces then $\mathbb{U}_1 \times \mathbb{U}_2$ is a Banach space too..

3. Let $C_b(I)$ be the linear space of all in $I \subset \mathbb{R}$ defined bounded functions $x(t)$ with $\|x\| = \sup_{t\in I} |x(t)|$. Prove that $(C_b(I), \|.\|)$ is a Banach space.

4. A subset A of a linear normed space $\mathbb{U}$ with $\|.\|$ is called convex, if for any $x, y \in A$ the "connection line" $\alpha x + (1 - \alpha)y$;
$\alpha \in (0, 1)$ belongs to A. Prove:
a) In a linear normed space the unit ball
$E = \{x \in \mathbb{U} \mid \|x\| \leq 1\}$ is a convex set.
b) The closure $\overline{A}$ of a convex set A is convex too.

5. Let $\mathbb{U} = C[a, b]$; $-\infty < a < b < \infty$ be a normed space with
$\|x\| = \max_{a\leq t\leq b} |x(t)|$.
Verify that:
a) $M = \{x \in \mathbb{U} \mid \int_a^b x(t)dt = 0\}$ is a closed subspace of $\mathbb{U}$. M is not dense in $\mathbb{U}$.
b) $M = \{x \in \mathbb{U} \mid x(a) = 1\}$ is closed and convex, but M is not a subspace of $\mathbb{U}$.
c) If φ is defined by $\varphi(x) = |x(a)|$ then φ is not a norm in $\mathbb{U}$.
d) If the norm is defined by $\|x(t)\|_1 = \int_a^b |x(t)|dt$, then $\|.\|_1$ is a norm in $\mathbb{U}$, but $\mathbb{U}$ is not a Banach space with respect to this norm. (advice: Construct a sequence of continuous functions which tends to a step function.)
e) The operators $A : \mathbb{U} \to \mathbb{R}$ and $B : \mathbb{U} \to \mathbb{U}$ are defined by

$$(Ax)(t) = x(a) \qquad \textit{and} \qquad (Bx)(t) = \int_a^t x(s)ds.$$

Prove:
A and B are linear continuous operators with $\|A\| = 1$ and $\|B\| = b - a$.
f) Verify that the operator $Fx = \int_0^1 sx(s)ds$ is continuous.
g) Look for numbers $\alpha \in \mathbb{R}$ such that the operator
$(Ax)(t) = \alpha \int_a^t \sin x(s)ds + 1$; $A : \mathbb{U} \to \mathbb{U}$ is contractive.

Seminar 6 / Pre-Hilbert Spaces and Hilbert Spaces

1. Let $p(t)$ be a continuous positive function defined in $[0,1]$. Prove:

$$(x,y) = \int_0^1 p(t)x(t)\overline{y(t)}dt, \qquad \forall x(t), y(t) \in C[0,1]$$

is an inner product in $C[0,1]$ (with the weight $p(t)$).

2. Verify that in any spaces with inner product the following statements are satisfied:
a) $x \perp y \Leftrightarrow \|x + \alpha y\| = \|x - \alpha y\| \quad \forall \alpha \in \mathbf{K}$
b) $x \perp y \Leftrightarrow \|x\| \leq \|x - \alpha y\| \quad \forall \alpha \in \mathbf{K}$

3. Orthonormize the system of functions $1, t, t^2, ...t^n$ in $L_2[0,\infty)$ with the weighted inner product

$$(f,g) = \int_0^\infty e^{-t} f(t)\overline{g(t)}dt.$$

4. Prove:
a) The Banach space $C[a,b]$; $-\infty < a < b < \infty$ with the maximum norm $\|x\| = \max_{a \leq t \leq b} |x(t)|$ is not a Hilbert space.
b) The Banach space $\mathbb{R}^n$ with $\|x\|_1 = \sum_{i=1}^n |x_i|$ is not a Hilbert space, but with $\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$ it is.

5. Prove that the unit ball is not compact in separable infinite dimensional Hilbert spaces.

6. Let $\mathbb{X} = C[-1;1]$ be a real space with the inner product

$$(x,y) = \int_{-1}^1 x(t)y(t)dt,$$

$E = \{x \in \mathbb{X} \mid x(t) = 0 \; ; \; t \leq 0\}, F = \{x \in \mathbb{X} \mid x(t) = 0 \; ; \; t \geq 0\}$.
Verify:
a) E, F are not closed subspaces of $\mathbb{X}$.
b) The set $E + F$ is not closed, too.

7. Calculate the best approximation of the function $f(t) = sint$ by the the polynoms $\phi_i(t) = t^{i-1}$; $i = 1, 2, ..5$ in the space $L_2[-\frac{\pi}{2}; \frac{\pi}{2}]$.